Reprinted from Nuclear Science and Engineering, Volume 6, No. 2, August 1959
Academic Press Inc. *Printed in U.S.A.*

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NUCLEAR SCIENCE AND ENGINEERING: **6**, 135–139 (1959)

Variational Calculation of Control Rod Reactivity

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Received December 12, 1958

The reactivity worth of cylindrical control rods of small cross section inserted in a cylindrical reactor is evaluated by means of a variational method. Both one-group and two-group diffusion theory are used and simple formulae are developed for a single eccentric and a concentric array of control rods. The results obtained are illustrated by means of a numerical example and compared to the results derived from other methods available in the literature.

NOMENCLATURE

A, A_i	constants
C, C_1	constants
D_1, D_2	diffusion coefficients
H	reactor height plus extrapolation lengths
J_i, I_i, K_i, Y_i	Bessel functions
L	diffusion length
M	migration length
R	reactor radius plus extrapolation length
m	number of control rods
k, k_0, k_∞	multiplication factors
λ	extrapolation length
μ_0, μ	eigenvalues
v_0	eigenvalue
ρ_i	reactivity
τ	age constant
ϕ, ϕ_1, ϕ_2	neutron fluxes
$\phi(r), \phi_1(r), \phi_2(r)$	neutron radial fluxes

INTRODUCTION

The purpose of this paper is to derive simple approximate formulae for the reactivity worth of cylindrical control rods inserted in bare, homogeneous thermal reactors by means of a variational method.

The control rods are assumed to have the same height as the reactor and to be fully inserted, their geometrical axis being parallel to the axis of the reactor.

First, the formulae for both a single eccentric and a concentric array of control rods are developed on the basis of one-group diffusion theory approximation assuming that the rods are black. Then, a two-group diffusion theory approximation is used and similar formulae are obtained assuming that the rods are black to thermal neutrons and transparent to fast neutrons.

The accuracy of the variational approach is finally illustrated by means of a numerical example, the results of which are compared to the results of other methods available in the literature.

ONE-GROUP APPROXIMATION

Eccentric Control Rod

According to one-group diffusion theory, the radial part of the flux $\phi(r)$ in a critical reactor of the type considered, without control rods, is the solution of the equation

$$\nabla_r^2\phi(r) + \mu_0^2\phi(r) = 0 \tag{1}$$

where

$$\mu_0^2 = (\beta_0/R)^2$$

and β_0 is defined by the smallest root of $J_0(\beta) = 0$, i.e., the first zero of the Bessel function J_0. Numerically, $\beta_0 = 2.405$.

The solution of Eq. (1) is

$$\phi_0(r) = J_0(\mu_0 r) \tag{2}$$

In case a black control rod of radius a is introduced in the reactor as shown in Fig. 1, the radial part of the flux under critical conditions is again a solution of Eq. (1) but for a different value μ of the eigen-

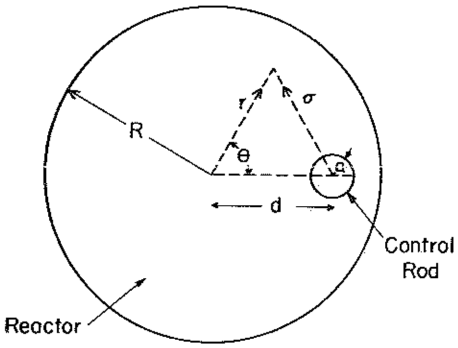

FIG. 1. Eccentric control rod arrangement

value μ_0. In order to determine the new eigenvalue by a variational method, a trial function is chosen which includes the singular part of the solution appearing when the control rod is introduced. The trial function considered here is

$$\phi(r) = J_0(\mu_0 r) + A Y_0(\mu_0 \sigma) \tag{3}$$

using the notation of the British Association Table of Bessel Functions.

The constant A is determined from the boundary condition that the logarithmic derivative of the flux $\phi(r)$ at the surface of the control rod be equal to the inverse extrapolation length. Thus A is approximately

$$A = -\frac{J_0(\mu_0 d)}{Y_0(\mu_0 a) + \lambda\mu_0 Y_1(\mu_0 a)} \tag{4}$$

However, the trial function (3) does not satisfy the boundary conditions at the outer surface of the reactor if the constant A is given by Eq. (4). Consequently the appropriate variational principle to be used is (*1*)

$$\mu^2 = -\frac{\int_v \phi(r)\nabla_r^2\phi(r)\, dv + \oint (\delta\phi/\delta n)\phi(r)\, ds}{\int_v \phi^2(r)\, dv} \tag{5}$$

The volume integral is to be taken over the volume of the reactor minus the rod and the surface integral over the outer surface of the reactor. $\delta\phi/\delta n$ denotes the normal gradient at the surface.

With obvious approximations Eq. (5) can be written as

$$\mu^2 = \mu_0^2 - \frac{\oint (\delta\phi/\delta n)\phi(r)\, ds}{\int_v \phi_0^2(r)\, dv} \tag{6}$$

assuming that the effect of the rod in the volume integrated flux is negligible.

In order to evaluate the surface integral, use is made of the addition theorem for Bessel functions, which, with the notation of Fig. 1, reads

$$Y_0(u_0\sigma) = Y_0(\mu_0 r)J_0(\mu_0 d) + 2\sum_{n=1}^{\infty} Y_n(\mu_0 r)J_n(\mu_0 d)\cos n\theta \tag{7}$$

Furthermore, using the fact that $\cos n\theta$ are orthogonal functions and the property of the derivative, $Y_n' = \frac{1}{2}(Y_{n-1} - Y_{n+1})$, Eq. (6) becomes

$$\mu^2 = \mu_0^2 + \frac{2\mu_0}{R} A J_0(\mu_0 d)\frac{Y_0(\beta_0)}{J_1(\beta_0)} \cdot \left[1 - \frac{A}{Y_0(\beta_0)J_1(\beta_0)J_0(\mu_0 d)}\sum_{n=0}^{\infty}\epsilon_n Y_n(\beta_0) \cdot [Y_{n-1}(\beta_0) - Y_{n+1}(\beta_0)]J_n^2(\mu_0 d)\right] \tag{8}$$

where

$$\epsilon_n = \begin{cases} \frac{1}{2} & n = 0 \\ 1 & n \geqq 1 \end{cases}$$

If the reactivity worth of the control rod is defined as

$$\rho = -(k - k_0)/k_\infty \tag{9}$$

and since

$$\mu_0^2 = \frac{k_0 - 1}{M^2} - \left(\frac{\pi}{H}\right)^2 \quad \mu^2 = \frac{k - 1}{M^2} - \left(\frac{\pi}{H}\right)^2 \tag{10}$$

it is found that

$$\rho = +0.818\frac{\mu_0 M^2}{k_\infty}\frac{J_0^2(\mu_0 d)}{Y_0(\mu_0 a) + \lambda\mu_0 Y_1(\mu_0 a)} \cdot \left[1 + \frac{3.78}{Y_0(\mu_0 a) + \lambda\mu_0 Y_1(\mu_0 a)}\sum_{n=0}^{\infty}\epsilon_n Y_n(\beta_0) \cdot [Y_{n-1}(\beta_0) - Y_{n+1}(\beta_0)]J_n^2(\mu_0 d)\right] \tag{11}$$

If the control rod is centrally located ($d = 0$) the reactivity worth given by Eq. (11) coincides (for small a) with the reactivity given by Glasstone and Edlund (*2*). Furthermore, it is seen that the reactivity is to first approximation proportional to the statistical weight $J_0^2(\mu_0 d)$.

The infinite series in Eq. (11) converges very quickly for reasonably small values of $\mu_0 d$. For instance, for $\mu_0 d = 1$, $[d = (r/2.405)]$, the first three terms approximate the sum with an error of less than 0.1 per cent.

Concentric Cylindrical Array of Rods

The case of a symmetrical array of m rods located on a cylindrical surface concentric with the reactor

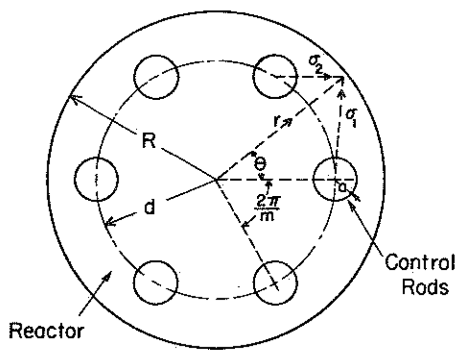

FIG. 2. Concentric array of control rods

is next considered (Fig. 2). The trial function to be used in the variational method is obtained by adding a singular solution for each control rod to the unperturbed solution (2). Consequently

$$\phi(r) = J_0(\mu_0 r) + A_1 \textstyle\sum_{i=1}^{m} Y_0(\mu_0 \sigma_i) \quad (12)$$

The constant A_1 is determined from a boundary condition similar to the one used in the case of a single eccentric rod. If the distance between the rods is not too small, the approximate value of A_1 is

$$A_1 = -\frac{J_0(\mu_0 d)}{Y_0(\mu_0 a) + \lambda\mu_0 Y_1(\mu_0 a) + \sum_{i=1}^{m-1} Y_0\left(\mu_0 2d \sin \frac{i\pi}{m}\right)} \quad (13)$$

Thus, using the addition theorem (7), the trial function becomes

$$\phi(r) = J_0(\mu_0 r) + mA_1 Y_0(\mu_0 r) J_0(\mu_0 d) + 2A_1 \sum_{i=1}^{m} \sum_{n=1}^{\infty} Y_n(\mu_0 r) J_n(\mu_0 d) \cos n\left(\theta - \frac{2i\pi}{m}\right) \quad (14)$$

or, performing the addition over i:

$$\phi(r) = J_0(\mu_0 r) + 2\, mA_1 \sum_{n=} \epsilon_n Y_n(\mu_0 r) J_n(\mu_0 d) \cos n\theta \quad (15)$$

where the summation runs only over integer multiples of m (zero included).

Using the same variational principle as in the case of a single eccentric rod, the reactivity worth of the array is found to be

$$\rho_1 = -0.818 \frac{\mu_0^2 M^2}{k_\infty} mA_1 J_0(\mu_0\, d) \cdot \left[1 - \frac{3.78 A_1}{J_0(\mu_0\, d)} \sum_{n=m} \epsilon_n \left[Y_n(\beta_0) - Y_{n-1}(\beta_0)\right] \cdot Y_{n+1}(\beta_0) J_n^2(\mu_0\, d)\right] \quad (16)$$

More complicated problems can be treated in the same way. In particular the procedure of dealing with several concentric arrays of control rods, with or without a central rod, is evident. However, it should always be kept in mind that the method fails if (a) the radius of the rods is too large, (b) the distance between the rods is too small, or (c) the over-all effect on the reactivity is too large.

The exact formulation of the validity of the approximation is extremely difficult because of the use of the variational method.

TWO-GROUP APPROXIMATION

ECCENTRIC CONTROL ROD

The fast and thermal fluxes of a two-group diffusion theory approximation for a critical reactor without control rods are solutions of the following equations:

$$\nabla^2\phi_1 - \frac{1}{\tau}\phi_1 + k\frac{D_2}{D_1}\frac{1}{L^2}\phi_2 = 0 \quad (17)$$

$$\nabla^2\phi_2 - \frac{1}{L^2}\phi_2 + \frac{D_1}{D_2}\frac{1}{\tau}\phi_1 = 0 \quad (18)$$

The radial parts $\phi_1(r)$ and $\phi_2(r)$ of the fluxes are linear combinations of the solutions of

$$\nabla_r^2 X(r) + \mu_0^2 X(r) = 0 \quad (19)$$

$$\nabla_r^2 Z(r) - \nu_0^2 Z(r) = 0 \quad (20)$$

where

$$\mu_0^2 = \left(\frac{\beta_0}{R}\right)^2 \qquad \nu_0^2 = b_2^2 + \left(\frac{\pi}{H}\right)^2$$
$$b_2^2 = b_1^2 + \frac{1}{\tau} + \frac{1}{L^2} \qquad b_1^2 = \left(\frac{\beta_0}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \quad (21)$$

and the value of k_0 for criticality is the solution of

$$b_1^2 = \frac{1}{2}\left[-\left(\frac{1}{\tau} + \frac{1}{L^2}\right) + \sqrt{\left(\frac{1}{\tau} + \frac{1}{L^2}\right)^2 + \frac{4(k_0 - 1)}{\tau L^2}}\,\right] \quad (22)$$

More specifically, $\phi_1(r)$ and $\phi_2(r)$ are

$$\phi_1(r) = X(r) + CZ(r) = J_0(\mu_0 r) \quad (23)$$

$$\phi_2(r) = S_1 X(r) + CS_2 Z(r) = S_1 J_0(\mu_0 r) \quad (24)$$

where

$$S_1 = \frac{D_1}{\tau D_2}\frac{L^2}{1 + b_1^2 L^2}$$
$$S_2 = -\frac{D_1}{D_2}\frac{1}{1 + b_1^2 \tau} \quad (25)$$

In case an eccentric control rod is introduced (Fig. 1) which is black to thermal neutrons and transparent to fast neutrons, Eq. (19) is still applicable, but for a different eigenvalue μ. $X(r)$ contains a singular solution and so does $Z(r)$ and may be assumed as

$$X(r) = J_0(\mu_0 r) + A_2 Y_0(\mu_0 \sigma) \tag{26}$$

$$Z(r) = K_0(v_0 \sigma) + C_1 I_0(v_0 r) \tag{27}$$

The constant A_2 is defined from the boundary conditions that the gradient of $\phi_1(r)$ vanishes and the logarithmic derivative of $\phi_2(r)$ is equal to the inverse extrapolation length at the surface of the control rod. If n is the inner normal at the rod surface, then the following equations apply at the rod boundary.

$$\frac{\delta\phi_1(r)}{\delta n} = \frac{\delta X(r)}{\delta n} + C\frac{\delta Z}{\delta n} = 0 \tag{28}$$

$$\phi_2(r) + \lambda\frac{\delta\phi_2}{\delta n} = S_1\left[X(r) + \lambda\frac{\delta X(r)}{\delta n}\right] + CS_2\left[Z(r) + \lambda\frac{\delta Z(r)}{\delta n}\right] = 0 \tag{29}$$

or, combining Eqs. (28) and (29)

$$S_1\frac{X(r)}{[\delta X(r)/\delta n]} = S_2\frac{Z(r)}{[\delta Z(r)/\delta n]} - \lambda(S_1 - S_2) \tag{30}$$

Since the radius of the rod is assumed small, the ratio $Z(r)/[\delta Z(r)/\delta n]$ can be approximated by the ratio $K_0(v_0 a)/v_0 K_1(v_0 a)$. Thus, Eq. (30) becomes

$$\left[\frac{S_2}{S_1}\frac{1}{v_0}\frac{K_0(v_0 a)}{K_1(v_0 a)} - \lambda\left(1 - \frac{S_2}{S_1}\right)\right]\cdot\frac{\delta X(r)}{\delta n} - X(r) = 0 \tag{31}$$

From Eq. (31) the constant A_2 is approximately

$$A_2 = -J_0(\mu_0 d)\Big/\left\{Y_0(\mu_0 a) + \frac{\tau}{L^2}\frac{1 + b_1^2 L^2}{1 + b_1^2\tau}\cdot\frac{\mu_0}{v_0}\frac{Y_1(\mu_0 a)}{K_1(v_0 a)}K_0(v_0 a) + \frac{D_1}{D_2}\cdot\left[1 + \frac{\tau}{L^2}\frac{1 + b_1^2 L^2}{1 + b_1^2\tau}\right]\lambda\mu_0 Y_1(\mu_0 a)\right\} \tag{32}$$

The variational principle is applied to Eq. (19). The boundary condition for $X(r)$ at the rod surface is given by Eq. (31). However, $X(r)$ does not vanish at the outer surface of the reactor. Consequently, the same variational principle previously used is again applicable and it is found that

$$\Delta b_1^2 = \Delta\mu^2 \approx \frac{k - k_0}{\tau L^2\sqrt{\left(\frac{1}{\tau} + \frac{1}{L^2}\right) + \frac{4(k_0 - 1)}{\tau L^2}}} = \frac{k - k_0}{\tau + L^2 + 2b_1^2\tau L^2} \tag{33}$$

$$\rho_2 = -0.818 \cdot \frac{\mu_0^2(\tau + L^2 + 2b_1^2\tau L^2)}{k_\infty} A_2 J_0(\mu_0 d) \cdot\left[1 - \frac{3.78A_2}{J_0(\mu_0 d)}\sum_{n=0}^{\infty}\epsilon_n Y_n(\beta_0)\cdot[Y_{n-1}(\beta_0) - Y_{n+1}(\beta_0)]J_n^2(\mu_0 d)\right] \tag{34}$$

Concentric Cylindrical Array of Rods

The geometrical arrangement is shown in Fig. 2. The procedure is similar to the one used above.

The trial function is

$$X(r) = J_0(\mu_0 r) + A_3\textstyle\sum_{i=1}^{m} Y_0(\mu_0\sigma_i) \tag{35}$$

The constant A_3 is defined by the boundary condition (31) and is

$$A_3 = -J_0(\mu_0 d)\Big/\left\{Y_0(\mu_0 a) + \sum_{i=1}^{m-1}\cdot Y_0\left(2\mu_0 d\sin\frac{i\pi}{m}\right) + \frac{\tau}{L^2}\frac{1 + b_1^2 L^2}{1 + b_1^2\tau}\frac{\mu_0}{v_0}\frac{Y_1(\mu_0 a)}{K_1(v_0 a)}\cdot K_0(v_0 a) + \frac{D_1}{D_2}\left[1 + \frac{\tau}{L^2}\frac{1 + b_1^2 L^2}{1 + b_1^2\tau}\right]\lambda\mu_0 Y_1(\mu_0 a)\right\} \tag{36}$$

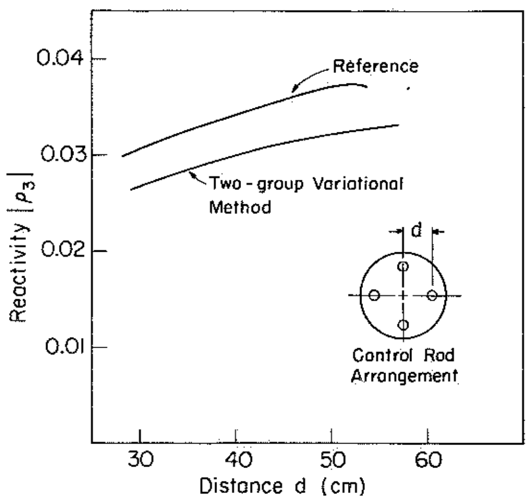

FIG. 3. Reactivity variation vs d.

After application of the variational principle the following expression for the reactivity results

$$\rho_3 = -0.818 \frac{\mu_0^2(\tau + L^2 + 2b_1^2\tau L^2)}{k_\infty} mA_3 J_0(\mu_0 d) \cdot \left[1 - \frac{3.78mA_3}{J_0(\mu_0 d)} \sum_{n=m} \epsilon_n Y_n(\beta_0) \cdot [Y_{n-1}(\beta_0) - Y_{n+1}(\beta_0)] J_n^2(\mu_0 d)\right] \quad (37)$$

NUMERICAL EXAMPLE

The reactivity ρ_3 for a group of four control rods is plotted in Fig. 3. The numerical data used are

$R = 205$ cm $k_\infty = 1.1558$
$H = 410$ cm $\lambda = 2.15$ cm
$D_1 = 1.01$ cm $L^2 = 188$ cm^2
$D_2 = 0.819$ cm $a = 4$ cm.
$\tau = 118$ cm^2

and are taken from Carlovik *et al.* (*3*). On the same figure the more exact results of Carlovik *et al.* are also plotted for comparison. The agreement is rather good (12 per cent).

For the case of 12 rods which is also treated in Ref. *3*, the method described in this paper gives completely meaningless reactivity values, which is to be expected, since many of the assumptions are not satisified.

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