Reprinted from INFORMATION AND CONTROL, Volume 6, No. 3, September 1963

INFORMATION AND CONTROL 6, 276–296 (1963)

Stability Criteria for a Class of Nonlinear Systems

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A general class of nonlinear systems is investigated from the standpoint of global asymptotic stability. It is shown that such stability of a unique equilibrium state is guaranteed if the system represented by the first approximation is either unconditionally stable or if the system of the first approximation is conditionally stable and the characteristic values (poles) lie in a limited region of the left-half complex plane.

The systems considered are generalizations of the systems proposed by Lur'e and Letov. The stability criteria derived by these authors are shown to be, in general, more restrictive than it is necessary to assure global asymptotic stability.

The generalization is achieved by replacing the discrete variables of multivariable autonomous dynamical systems by a parametrized continuum. The introduction of the continuum both broadens the class of systems under consideration and expedites the investigation of the problem of stability.

I. INTRODUCTION

The purpose of this paper is to present tractable criteria for global asymptotic stability of a broad class of autonomous dynamical systems with elements of arbitrary nonlinear characteristics. The characteristics are defined only to the extent of their belonging to a certain class of functions.

It is shown that, if a system of the class is to be asymptotically stable for perturbations of any magnitude, the condition is that either the system represented by the first approximation is unconditionally stable or that it is conditionally stable and the characteristic values (poles) lie in a limited region of the left half complex plane. The results are established by means of Liapunov's direct method and an integral formulation of the problem.

Special cases of the problems considered have been analyzed by Lur'e

(1951), Letov (1961), and other authors in the USSR by means of Liapunov's direct method. All these authors, however, derived conditions for global asymptotic stability which, in general, are either more restrictive than the conditions required for the system represented by the first approximation to be stable or in a form that apparently is not tractable.

The paper is organized as follows: first, a brief discussion of Liapunov's direct method is included for convenience; second, a brief account of the types of problems treated by Lur'e and Letov is given and their conclusions are summarized as background information for the generalized problem and approach that are discussed in the following section; third, a more general class of problems than that proposed by Lur'e is considered and conditions for global asymptotic stability are derived.

II. STABILITY BY LIAPUNOV'S DIRECT METHOD

A. The Problem

An autonomous dynamical system, be it electrical, mechanical, nuclear etc. or any combination of these, can be described by a certain number of n variables $x_1, x_2, \cdots x_n$. These variables may be visualized as the coordinates of a point M in a n-dimensional space or equivalently, as the components of a vector x in the same space. Each point or vector represents a state of the system.

Without loss of generality it may be assumed that the point $x = 0$ is an equilibrium state. One of the fundamental questions of the theory of control is the type of stability of the equilibrium state. Specifically, if at time $t = 0$ the system is perturbed from its equilibrium state ($x(t = 0) \neq 0$) the question arises as to whether for $t \to \infty$ the variables of the system resume their equilibrium values (the system is asymptotically stable), are bounded (the system is stable), or diverge (the system is unstable).

This question can be elegantly answered by means of Liapunov's direct method, if the dynamics of the system are adequately represented by a set of ordinary nonlinear differential equations. More precisely, assume that the dynamic behavior of an autonomous system[1] is represented by the set of n ordinary differential equations:

$$\frac{dx_k}{dt} = \dot{x}_k = X_k(x_1, x_2, \cdots x_n) \qquad k = 1, 2, \cdots n \quad (1)$$

[1] Similar procedures have been developed for nonautonomous systems (Malkin, 1950).

or by the equivalent vector equation

$$\dot{x} = X(x) \tag{2}$$

where X_k is a nonlinear function of $x_1, x_2, \cdots x_n$. In addition, suppose that $x = 0$ is an equilibrium state, i.e. $X(0) = 0$. The type of stability of this state, for different initial perturbations, can be investigated without integration of Eq. (2) by the method described in the following section.

B. Liapunov's Direct Method

Liapunov's direct method of stability is based on the existence of a positive definite scalar function $V(x)$ with the following properties:

a. $V(x)$ is continuous together with its first partial derivatives in a certain open region Ω about the origin $x = 0$.

b. $V(0) = 0$, $V(\infty) = \infty$.

c. Outside the origin and always in Ω, $V(x)$ is positive. In other words, the origin is an isolated minimum of $V(x)$.

d. If $\dot{V}(x) \leqq 0$ (subject to Eq. (2)), $V(x)$ is called a Liapunov function.

With these definitions, Liapunov's main stability theorems may be summarized as follows:

I. Stability Theorem. *If there exists in some neighborhood of the origin a Liapunov function $V(x)$, then the origin is stable for all perturbations lying in Ω.*

II. Asymptotic Stability Theorem. *If, in addition to the requirements of Theorem I, $\dot{V}(x)$ is negative definite, then the stability is asymptotic.*

III. Instability Theorem. *If $V(x) > 0$, $V(0) = 0$, and $\dot{V}(x) > 0$ in Ω, then the origin is unstable.*

There are other variations and generalizations of Liapunov's theorems. For these, however, as well as for the proof and geometric interpretation of Theorems I–III, the reader is referred to the literature (LaSalle and Lefschetz, 1961; Lefschetz, 1957). Suffice it to note only that the existence of a Liapunov function guarantees the stability of the origin or what has been assumed as the equilibrium state of the system described by Eq. (2).

It must be also emphasized that stability or asymptotic stability of an equilibrium state of a physical system does not necessarily imply the existence of Liapunov functions. However, from a practical standpoint, this is not important. A particular Liapunov function yields certain sufficient requirements for stability which is what is desired in practice. Of course, it is also essential to select the Liapunov function which results in the least restrictive requirements possible. In fact, if it can be shown that the requirements are such that they cannot be further improved, then the corresponding Liapunov function yields both necessary and sufficient conditions for stability.

Having available a technique to investigate the stability of equilibrium states, the next question is: "Given a specific system of equations, how does one construct a Liapunov function?" This problem has been dealt with by many authors. In particular, Lur'e has developed a procedure which is the subject of discussion of the next section.

III. LUR'E'S METHOD

The purpose of this section is to summarize Lur'e's method for the construction of Liapunov functions for a large class of nonlinear systems which are representative of many practical control problems.

Specifically, consider control systems whose dynamic equations are:

$$\dot{x}_k = \sum_{\alpha=1}^{m} b_{k\alpha} x_\alpha + n_k \mu; \; k = 1, 2, \cdots m \tag{3}$$

where $x_1, \cdots x_m$ are the system variables, μ is the coordinate of the regulating organ, and $b_{k\alpha}$, n_k are constant coefficients. The coordinate μ obeys the equation

$$\begin{aligned} V^2\ddot{\mu} + \dot{\mu} + S\mu &= f(\sigma) \\ \sigma = \sum_{\alpha=1}^{m} p_\alpha x_\alpha - r\mu; &\qquad r > 0 \end{aligned} \tag{4}$$

where V^2, S, p_α, r are constants and $f(\sigma)$ belongs to either of the following two classes of functions:

Class (A) $\qquad f(\sigma) = 0 \quad \text{for} \quad |\sigma| \leqq \sigma^*$

$$\sigma f(\sigma) > 0 \quad \text{for} \quad |\sigma| > \sigma^*$$

Class (A_1) $\qquad \sigma^* = 0; \qquad \left.\frac{df}{d\sigma}\right|_{\sigma=0} = h \geqq 0$

$$\sigma\phi(\sigma) > 0; \qquad \phi(\sigma) = f(\sigma) - h\sigma$$

Assume that the only equilibrium state of the system is $x_1 = x_2 = \cdots x_m = \mu = \sigma = 0$. The construction of a Liapunov function, for the study of the stability of this state is greatly facilitated if the system of equations (3) and (4) is reduced to a canonical form. The reduction can be achieved by an infinite variety of linear transformations of the variables x_k. To see this, consider the following two cases:

a. Assume that all the characteristic roots ρ_k of the $m \times m$ matrix

$$B = [b_{k\alpha}] \tag{5}$$

namely, the roots of the determinantal equation

$$|B - \rho I| = 0 \tag{6}$$

where I is the unit or identity matrix, are distinct and have the property that Re $\rho_k < 0$. In other words, assume that the system, with the regulating organ disconnected, is inherently stable. Thus, admit with Lur′e that Eqs. (3) and (4) can be reduced into the canonical form

$$\begin{aligned} \dot{x}_k &= \rho_k x_k + f(\sigma); \qquad k = 1, 2, \cdots n; \qquad n = m + 2 \\ \dot{\sigma} &= \sum_{k=1}^{n} \beta_k x_k - r f(\sigma); \qquad r > 0 \\ \sigma &= \sum_{k=1}^{n} \gamma_k x_k \end{aligned} \tag{7}$$

where

x_k	$(k = 1, 2, \cdots m)$ is used again to denote the new variables
x_{m+1}, x_{m+2}	are two variables that reduce the second order equation for μ to two first order equations
β_k, γ_k	are constants derived from the original coefficients
ρ_{m+1}, ρ_{m+2}	are the roots of the equation $V^2\rho^2 + \rho + S = 0$ and are presumed such that that Re ρ_{m+1}, Re $\rho_{m+2} < 0$.

Lur′e (1951) gives explicit formulas for the transformation matrix and the coefficients β_k, γ_k. A block diagram of the system represented by this canonical form is shown in Fig. 1.

A simple Liapunov function can now be constructed for the canonical system (7). To prove this assertion, assume that among the n characteristic roots ρ_k there are (s) real $(\rho_1, \cdots \rho_s)$ and $(n - s)/2$ complex conjugate pairs $(\rho_{s+1}, \cdots \rho_n)$.[2] Consider the function

$$V = -\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{a_1 a_k}{\rho_1 + \rho_k} x_i x_k + \int_0^{\sigma} f(\sigma)\, d\sigma \tag{8}$$

[2] It is evident that the corresponding canonical variables x_k are similar in character.

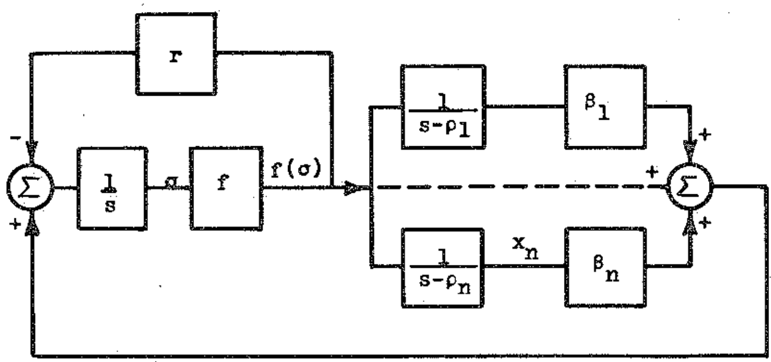

FIG. 1. Block diagram representation of the nonlinear system represented by Eqs. (7). In the first approximation the zero memory nonlinear function $f(\sigma)$ is replaced by a pure positive gain h. For $h = 0$ the first approximation is assumed inherently stable, i.e., Re $\rho_k < 0$ for $k = 1, 2 \cdots n$.

where $a_i, \cdots a_s$ are arbitrary real numbers and $a_{s+1}, \cdots a_n$ are arbitrary complex conjugate pairs. Notice that the first term in (8) is a positive definite form and that it vanishes only at the origin because:

$$\begin{aligned} -\frac{1}{\rho_i + \rho_k} &= \int_0^\infty e^{(\rho_i+\rho_k)\tau}\, d\tau - \sum_{i=1}^n \sum_{k=1}^n \frac{a_i\, a_k}{\rho_i + \rho_k}\, x_i\, x_k \\ &= \int_0^\infty \left[\sum_{k=1}^n a_k\, x_k\, e^{\rho_k \tau}\right]^2 d\tau \end{aligned} \tag{9}$$

Consequently, V is a positive definite form with continuous partial derivatives when $f(\sigma)$ is of class (A) or (A_1), and furthermore V grows indefinitely as x and σ grow indefinitely.

The time derivative of V is:

$$\begin{aligned} \dot{V} = &-\left[\sum_{k=1}^n a_k\, x_k + \sqrt{r} f(\sigma)\right]^2 \\ &+ f(\sigma) \sum_{k=1}^n \left[\beta_k + 2\sqrt{r} a_k - 2a_k \sum_{i=1}^n \frac{a_i}{\rho_k + \rho_i}\right] x_k \end{aligned} \tag{10}$$

This time derivative would be negative definite if:

$$\beta_k + 2\sqrt{r} a_k - 2a_k \sum_{i=1}^n \frac{a_i}{\rho_k + \rho_i} = 0; \qquad k = 1, 2, \cdots n \tag{11}$$

According to Liapunov's direct method, when conditions (11) are satisfied the system described by Eqs. (3) and (4) is asymptotically stable to all perturbations.

Equations (11) may be interpreted as sufficient restrictions on the characteristic roots ρ_k and the control coefficients β_k for the system to be globally asymptotically stable. These restrictions result from the requirement that Eqs. (11) must admit solutions $a_1, \cdots a_s$ that are real and solutions $a_{s+1}, \cdots a_n$ that are complex conjugate pairs. They can be written explicitly in terms of the ρ_k's and the β_k's as it is shown in Section IV.

This completes the discussion of the first case of Lur'e's approach to the problem of global asymptotic stability.

Actually, Letov (1961) presents other variations of Lur'e's method. He shows, however, that in special simple examples all these variations lead to more restrictive requirements than those implied by Eqs. (11).

b. If the uncontrolled system is inherently unstable, assume that the characteristic roots r_k of the modified matrix

$$B_1 = \left[b_{k\alpha} + \frac{n_k\, p_\alpha}{r} \right] \tag{12}$$

are distinct and have the property that Re $r_k < 0$. Thus, admit with Lur'e that Eqs. (3) and (4) can be reduced to the canonical form

$$\begin{aligned} \dot{x}_k &= r_k x_k + \sigma \qquad\qquad k = 1, 2, \cdots m \\ \dot{\sigma} &= \sum_{k=1}^{m} \bar{\beta}_k x_k - \bar{p}\sigma - f(\sigma) \end{aligned} \tag{13}$$

provided that $V^2 = 0$ (Eq. 4).[3] The existence of the transformation matrix and explicit formulas for the coefficients $\bar{\beta}_k$ and $\bar{p}$ are given by Letov (1961).

Following arguments and assumptions similar to those used for case (a), notice that the positive definite function

$$V = -\sum_{i=1}^{m} \sum_{k=1}^{m} \frac{a_i\, a_k}{r_i + r_k}\, x_i\, x_k + \frac{1}{2}\sigma^2 \tag{14}$$

admits a negative definite derivative:

$$\dot{V} = -\left[\sum_{k=1}^{m} a_k\, x_k + \sqrt{\bar{p}}\sigma \right]^2 - \sigma f(\sigma) \tag{15}$$

[3] The assumption $V^2 = 0$ does not imply any restriction whatsoever on the class of systems under consideration.

when $\bar{\rho} > 0$ and

$$\bar{\beta}_k + 2\sqrt{\bar{\rho}}a_k - 2a_k \sum_{i=1}^{m} \frac{a_i}{r_k + r_i} = 0; \qquad k = 1, 2, \cdots m \quad (16)$$

or, a negative definite derivative:

$$\dot{V} = -\left[\sum_{k=1}^{m} a_k x_k + \sqrt{\bar{\rho} + h}\sigma\right]^2 - \sigma\phi(\sigma) \quad (17)$$

when $\bar{\rho} + h > 0$, $(f(\sigma) = h\sigma + \phi(\sigma))$, and

$$\bar{\beta}_k + 2\sqrt{\bar{\rho} + h}a_k - 2a_k \sum_{i=1}^{m} \frac{a_i}{r_k + r_i} = 0; \qquad k = 1, 2, \cdots m \quad (18)$$

Equations (16) or (18) play the same role as Eqs. (11). In other words, the requirement that $a_1, \cdots a_s$ be real and $a_{s+1}, \cdots a_m$ be complex conjugate pairs results in a set of conditions for the $\bar{\beta}_k$'s and the r_k's which are sufficient to guarantee the global asymptotic stability of Eqs. (13).

Conditions (16) and (18) are slightly different from those presented by Letov (1961). As it will become evident, however, from the discussion of the next section, all of Letov's conditions are unnecessarily more restrictive for this canonical form.

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBAL ASYMPTOTIC STABILITY

A. General Remarks

The purpose of this section is to generalize Lur'e's method to a broader class of nonlinear systems and to investigate the conditions for which asymptotic stability in the small is sufficient to guarantee global asymptotic stability.

To this effect, the set of discrete variables $x_k(t)$ is replaced by a continuum $x(\rho, t)$, where ρ is a complex parameter with values only throughout the left-half complex plane (Re $\rho < 0$) and $x(\rho, t)$ is an analytic function of ρ, real for ρ real and complex for ρ complex. The summations in the equations for the real variable σ are replaced by contour Stieltjes integrals, taken over the entire left-half complex plane. Thus, systems are considered whose dynamics are assumed to be describable by the set of equations:

$$\begin{aligned} \frac{d}{dt} x(\rho, t) &= \rho x(\rho, t) + f(\sigma) \\ \frac{d}{dt} \sigma &= \oint x(\rho, t)\, d\beta(\rho) - rf(\sigma) \end{aligned} \tag{19}$$

or

$$\begin{aligned} \frac{d}{dt} x(\rho, t) &= \rho x(\rho, t) + \sigma \\ \frac{d}{dt} \sigma &= \oint x(\rho, t)\, d\beta_1(\rho) - \rho_1 \sigma - f(\sigma) \end{aligned} \tag{20}$$

The functions $\beta(\rho)$ and $\beta_1(\rho)$ have all their singularities in the left-half complex plane and it is assumed that the contour integrals yield real functions of t for all t. The coefficients r, ρ_1 are constant.

It is readily verified that Eqs. (7) or (13) are special cases of Eqs. (19) or (20) respectively. For example, if:

$$d\beta(\rho) = \frac{1}{2\pi j}\left[\sum_{k=1}^{n} \frac{\beta_k}{\rho - \rho_k}\right] d\rho; \qquad \text{Re } \rho_k < 0 \tag{21}$$

then it is found that:

$$\oint x(\rho, t)\, d\beta(\rho) = \sum_{k=1}^{n} \beta_k\, x_k(\rho_k, t) = \sum_{k=1}^{n} \beta_k\, x_k \tag{22}$$

Without loss of generality it may again be assumed that the state $x = \sigma = 0$ is the equilibrium state for either system (19) or (20).[4] The question of global stability for this state can be answered by a straightforward generalization of Lur'e's method. Before this generalization is presented, however, it is instructive to consider the behavior of the first approximation of Eqs. (19) and (20).

B. Stability of the First Approximation

Consider the system of Eqs. (19). The first approximation is[5]:

[4] The implication of the existence of a unique equilibrium state is that:

$$\oint \frac{d\beta(\rho)}{\rho} + r > 0 \quad \text{or} \quad \oint \frac{d\beta_1(\rho)}{\rho} + \rho_1 + h > 0$$

[5] Note that for $h = 0$, the first approximation is, by definition, stable.

$$\begin{aligned} \frac{d}{dt} x(\rho, t) &= \rho x(\rho, t) + h\sigma; \qquad f(\sigma) \sim h\sigma \\ \frac{d}{dt} \sigma &= \oint x(\rho, t)\, d\beta(\rho) - rh\sigma \end{aligned} \tag{23}$$

Define the function $g(t)$ such that

$$\begin{aligned} g(t) &= \oint e^{\rho t}\, d\beta(\rho) \qquad t > 0 \\ &= 0 \qquad\qquad\qquad t \leqq 0 \end{aligned} \tag{24}$$

The Laplace transform of σ is

$$\bar{\sigma} = \frac{\sigma(0) + G_0(s)}{s\left[1 + h\dfrac{r - G(s)}{s}\right]} \tag{25}$$

where $\sigma(0)$ is the initial value of σ, $G_0(s)$ is the Laplace transform of $\oint x(\rho, 0)e^{\rho t}\, d\beta(\rho)$, $x(\rho, 0)$ is the initial value of $x(\rho, t)$, and $G(s)$ is the Laplace transform of $g(t)$. If the system is to be unconditionally stable[6] for all members of the class of functions $f(\sigma)$, namely for all values of $h \geqq 0$, then from linear feedback theory it is concluded that the necessary and sufficient condition is that $(r - G(s))$ be a positive real function[7] without any roots on the $j\omega$-axis. In other words, unconditional stability is achieved if and only if

$$r - \int_0^\infty g(t) \cos \omega t\, dt > 0 \tag{26}$$

On the other hand, if the class of functions $f(\sigma)$ is restricted to a limited range of values of h, then the necessary and sufficient condition for stability of the system of the first approximation (Eqs. (23)) is that the function

$$F(s) = s + h(r - G(s)) \tag{27}$$

[6] Unconditionally stable is used here in the accepted sense of linear feedback theory.

[7] A function $F(s)$ is positive real if $F(s)$ is real for s real and Re $F(s) \geqq 0$ for Re $s \geqq 0$. This definition is equivalent to requiring that Re $F(j\omega) \geqq 0$ (Guillemin, 1951). The equality sign is applicable only when there are roots on the $j\omega$-axis.

has all its zeros in the left-half complex plane. This requirement may be satisfied even when $(r - G(s))$ is not a positive real function.

Similarly, it can be readily concluded that the necessary and sufficient condition for the first approximation of Eqs. (20) to be unconditionally stable for all $h \geqq 0$ is that $\rho_1 > 0$ and that the function $(\rho_1 - G_1(s))$ be positive real, where $G_1(s)$ is the Laplace transform of $g_1(t) = \oint e^{\rho t}\, d\beta_1(\rho)$ for $t > 0$ and $g_1(t) = 0$ for $t \leqq 0$. If $\rho_1 < 0$, then the necessary and sufficient conditions for the first approximation to be stable are $\rho_1 + h > 0$ and that the function

$$F_1(s) = s + \rho_1 + h - G_1(s) \tag{28}$$

admits roots only in the left-half complex plane. Notice that Eq. (28) is satisfied for all admissible values of h if and only if $(-G(s))$ is a positive real function without $j\omega$-axis zeros.

C. Global Asymptotic Stability

Consider next the question of global asymptotic stability of systems represented by Eqs. (19). To this end, define the function:

$$V = \int_0^\infty \left| \oint a(\rho)x(\rho, t)e^{\rho s}\, d\rho \right|^2 ds + \int_0^\sigma f(\sigma)\, d\sigma \tag{29}$$

where the contour integral is taken over the left-half complex plane. It is evident that if $a(\rho)$ is an arbitrary real meromorphic function with singularities in the left-half complex plane, then V is a positive definite function of the type specified by Liapunov. The positive definite character of V results from the fact that the first term in the right hand side is positive definite according to Cauchy's residue theorem and the second is nonnegative. The time derivative of V is:

$$\begin{aligned}\dot{V} = -\left| \oint a(\rho)x(\rho, t)\, d\rho \right|^2 - rf^2(\sigma) \\ + f(\sigma)\Bigg[-\oint\oint \frac{a(\rho)a(\rho^*)}{\rho + \rho^*}\, x(\rho, t)\, d\rho\, d\rho^* \\ -\oint\oint \frac{a(\rho)a(\rho^*)}{\rho + \rho^*}\, x(\rho^*, t)\, d\rho\, d\rho^* \\ + \frac{1}{2}\oint x(\rho, t)\, d\beta(\rho) + \frac{1}{2}\oint x(\rho^*, t)\, d\beta(\rho^*)\Bigg]\end{aligned} \tag{30}$$

because $\oint x(\rho, t)\, d\beta(\rho) = \oint x(\rho^*, t)\, d\beta(\rho^*)$, where ρ^* is the complex conjugate of ρ. This derivative reduces to:

$$\dot{V} = -\left| \oint a(\rho)x(\rho, t)\, d\rho + \sqrt{r}f(\sigma) \right|^2 \tag{31}$$

if the function $a(\rho)$ satisfies the condition

$$\frac{d\beta(\rho)}{d\rho} - 2\oint \frac{a(\rho)a(\rho^*)}{\rho + \rho^*}\, d\rho^* = -2\sqrt{r}a(\rho) \tag{32}$$

Thus the question of global asymptotic stability of systems represented by (19) reduces to the establishment of the necessary and sufficient requirements for Eq. (32) to admit a function $a(\rho)$ as a solution and $\dot{V}$ to be negative definite. Notice that, for discrete variables, Eqs. (29) and (32) reduce to Eqs. (8) and (11), respectively.

Equation (32) admits a function $a(\rho)$, of the specified type, as a solution when $(r - G(s))$ is positive real. Indeed, Eq. (32) can be written as:

$$\begin{aligned} -g(t) &= -\oint e^{\rho t}\, d\beta(\rho) \\ &= 2\sqrt{r}\oint a(\rho)e^{\rho t}\, d\rho - 2\oint\oint \frac{a(\rho)a(\rho^*)}{\rho + \rho^*} e^{\rho t}\, d\rho\, d\rho^* \\ &= \sqrt{r}\oint a(\rho)e^{\rho t}\, d\rho + \sqrt{r}\oint a(\rho^*)e^{\rho^* t}\, d\rho^* \\ &\quad - \oint\oint \frac{a(\rho)a(\rho^*)}{\rho + \rho^*}(e^{\rho t} + e^{\rho^* t})\, d\rho\, d\rho^* \end{aligned} \tag{33}$$

If r is added to the cosine transform of $-g(t)$, it is found that:

$$\begin{aligned} r - \int_0^\infty g(t) \cos \omega t\, dt &= \left| \int_{-j\infty}^{j\infty} \frac{\rho a(\rho)}{\rho^2 + \omega^2}\, d\rho + \sqrt{r} \right|^2 \\ &\quad + \left| \int_{-j\infty}^{j\infty} \frac{\omega a(\rho)}{\rho^2 + \omega^2}\, d\rho \right|^2 \end{aligned} \tag{34}$$

where the contour integrals have been replaced by line integrals along the imaginary axis. For Eq. (34) to be satisfied, it is necessary that the function $(r - G(s))$ be positive real. This is indeed the case when the first approximation equations are unconditionally stable for all $h \geqq 0$.

Actually, the positive real character of $(r - G(s))$ is also sufficient. To prove this assertion proceed as follows. Since the function $a(\rho)$ is arbitrary, assume that $(a(\rho) + \sqrt{r}) = A(\rho)$ is a positive real function. Thus, the real part of $A(\rho)$ is an even function and the imaginary part is an odd function of frequency along the imaginary axis. The two terms in the right hand side of Eq. (34) can be interpreted as Hilbert transforms (Bode, 1952). In this manner, Eq. (34) can be written as:

$$r - \int_0^\infty g(t) \cos \omega t \, dt = [\text{Re } A(j\omega)]^2 + [\text{Im } A(j\omega)]^2 = | A(j\omega) |^2 \quad (35)$$

The meaning of Eq. (35) is that the square of the magnitude of the positive real function $A(\rho)$, along the imaginary axis, is equal to the real part of $(r - G(s))$ for $s = j\omega$. It is well known that given the magnitude of a positive real function along the imaginary axis it is always possible to find the function (Bode, 1952). Consequently, the condition that $(r - G(s))$ be a positive real function is also sufficient to assure the existence of $a(\rho)$. In addition, note that both $a(\rho)$ and $A(\rho)$ have the same singularities as $d\beta(\rho)$ and $A(\rho)$ has no zeros along the $j\omega$-axis.

Next, consider whether $\dot{V}$ is negative definite or not. To this end, assume that there are values of $x(\rho, t)$, $\sigma(t) \neq 0$ for which $\dot{V} = 0$. Thus, find from Eq. (31) that

$$f(\sigma(t)) = -\frac{1}{\sqrt{r}} \oint a(\rho)x(\rho, t) \, d\rho \quad (36)$$

and from Eq. (19) that

$$x(\rho, t) = \int_0^t f(\sigma(\tau))e^{\rho(t-\tau)} \, d\tau + x(\rho, 0)e^{\rho t} \quad (37)$$

Combination of Eqs. (36) and (37) yields

$$\int_0^t d\tau f(\sigma(\tau)) \oint d\rho[a(\rho)e^{\rho(t-\tau)} + \sqrt{r}\delta(t - \tau)] = -\oint a(\rho)x(\rho, 0)e^{\rho t} \, d\rho \quad (38)$$

The Laplace transform of Eq. (38) is

$$\overline{f(\sigma(t))}A(s) = -\overline{\oint a(\rho)x(\rho, 0)e^{\rho t} \, d\rho} \quad (39)$$

Since the singularities of $a(\rho)$ are all in the left-half complex plane and

$A(s)$ is positive real without zeros on the $j\omega$-axis, Eq. (39) admits a solution $f(\sigma(t))$ or $\sigma(t)$ that converges to a constant. The second of Eqs. (39) and the assumption about the existence of only one equilibrium state imply that the only admissible constant is $f(\sigma) = \sigma = 0$ and $x = 0$. Consequently, $\dot{V}$ is negative definite.

In summary, it is shown that a Liapunov function exists, which guarantees the global asymptotic stability of systems represented by (19), if $(r - G(s))$ is a positive real function. Since the same requirement is necessary and sufficient for the first approximation to be unconditionally stable for all members of the class of functions $f(\sigma)$, it is concluded that: The necessary and sufficient condition for system (19) to be asymptotically stable to all perturbations and for all $f(\sigma)$ is that the first approximation be unconditionally stable for all $h \geqq 0$.

By use of the same technique it can be generally shown that all forms of stability criteria that have been proposed by Lur′e and Letov (1961) for inherently stable systems yield at best equivalent or unnecessarily more restrictive conditions for the function $G(s)$, i.e., for the parameters β_k $(k = 1, 2, \cdots n)$ of the controller and the characteristic roots ρ_k.

For example, Letov considers another positive function

$$V = -\sum_{i=1}^{n}\sum_{k=1}^{n} \frac{a_i\, a_k}{\rho_i + \rho_k}\, x_i\, x_k + \int_0^{\sigma} f(\sigma)\, d\sigma + \frac{1}{2}\sum_{i=1}^{s} B_i\, x_i^2$$
$$+ \frac{1}{2}(B_{s+1} + B_{s+2})x_{s+1}\, x_{s+2} + \cdots + \frac{1}{2}(B_{n-1} + B_n)x_{n-1}\, x_n \tag{40}$$

where $B_i > 0$ for $i = 1, 2, \cdots n$ and $B_{s+1} = B_{s+2} \cdots B_{n-1} = B_n$ for all complex conjugate pairs $(x_{s+1}, x_{s+2}) \cdots (x_{n-1}, x_n)$. In terms of the continuum this function can be written as

$$V = \int_0^{\infty} \left| \oint a(\rho)x(\rho, t)e^{\rho s}\, d\rho \right|^2 ds$$
$$+ \int_0^{\sigma} f(\sigma)\, d\sigma + 2\int_{-j\infty}^{j\infty} |\, x(\rho, t)\,|^2\, dB(\rho) \tag{41}$$

where $dB(j\omega) > 0$ and all other quantities have the same definitions as before. Use of the previous techniques reveals that $\dot{V} < 0$ if

$$r - \int_0^{\infty} g(t)\cos\omega t\, dt - \int_{-j\infty}^{j\infty} \frac{dB(\rho)}{\rho^2 + \omega^2}$$
$$= \left| \int_{-j\infty}^{j\infty} \frac{\rho a(\rho)}{\rho^2 + \omega^2} + \sqrt{r} \right|^2 + \left| \int_{-j\infty}^{j\infty} \frac{\omega a(\rho)}{\rho^2 + \omega^2}\, d\rho \right|^2 \tag{42}$$

The meaning of Eq. (42) is that given a $dB(j\omega) > 0$ the function $a(\rho)$ exists if $(r - \int_0^\infty g(t) \cos \omega t \, dt)$ is more positive than it is necessary for Eq. (34) to be satisfied and hence for the linear approximation to be unconditionally stable. Or, if $a(\rho)$ is taken identically zero then $(-G(s))$ rather than $(r - G(s))$ must be positive real which is again more restrictive than necessary for unconditional linear stability.

The important implication of this presentation is that if the nonlinear systems (19) are to be designed for all members of the class of functions $f(\sigma)$ then the stability question can be investigated by considering only the equations of the first approximation.

Regarding the global asymptotic stability of systems represented by Eqs. (20), a procedure similar to the one used for Eqs. (19) yields that in general the sufficient condition is that $(\rho_1 + h - G(s))$ be positive real. This condition reduces to the necessary and sufficient requirement for the first approximation to be unconditionally stable for all $h \geqq 0$ when $\rho_1 > 0$ and it is more restrictive than the requirement implied by Eq. (28) when $\rho_1 < 0$, $\rho_1 + h > 0$.

Again, however, it can be readily verified that Lur'e's and Letov's results, for this class of systems, are unnecessarily more restrictive than the ones derived here. The verification is achieved by recasting Lur'e's and Letov's V-functions in terms of the continuum $x(\rho, t)$ and proceeding as above.

D. An Alternative Approach to the Problem of Stability

The distinction of the systems under consideration as being inherently stable or inherently unstable and therefore reducible to either of the canonical forms (7) or (13) and (19) or (20) is somewhat artificial. In general, it can be assumed that these systems are reducible into the form

$$\frac{d}{dt} x(\rho, t) = \rho x(\rho, t) + \sigma \tag{43a}$$

$$\frac{d}{dt} \sigma = \oint x(\rho, t) \, d\beta_1(\rho) - \rho_1 \sigma - h\sigma - \phi(\sigma) \tag{43b}$$

where ρ and the singularities of $d\beta_1(\rho)$ are restricted to be in the left-half complex plane.

As already indicated, the necessity for stability of the first approximation implies that $F_1(s)$ (Eq. 28) has all its zeros or that $1/F_1(s)$ has all its singularities in the left half complex plane, respectively.

The investigation of global asymptotic stability is facilitated if Eqs. (43) are recast into an integral form. To this effect, note that Eq. (43a) is linear and can be readily integrated to yield:

$$x(\rho, t) = \int_0^t \sigma(\tau) e^{\rho(t-\tau)}\, d\tau + x(\rho, 0) e^{\rho t} \tag{44}$$

Thus, Eq. (43b) reduces to:

$$\frac{d\sigma}{dt} = \oint d\beta_1(\rho) \int_0^t d\tau \sigma(\tau) e^{\rho(t-\tau)} - (\rho_1 + h)\sigma + \oint x(\rho, 0) e^{\rho t}\, d\beta_1(\rho) - \phi(\sigma) \tag{45}$$

The Laplace transform of Eq. (45) is

$$\bar{\sigma}[s + \rho_1 + h - G_1(s)] = \bar{\sigma} F_1(s) = G_2(s) - \overline{\phi(\sigma)} \tag{46}$$

where $G_2(s)$ is the Laplace transform of:

$$\begin{aligned} g_2(t) &= \oint x(\rho, 0) e^{\rho t}\, d\beta_1(\rho) + \sigma(0) \qquad && t > 0 \\ &= 0 && t \leqq 0 \end{aligned} \tag{47}$$

The inverse transform of Eq. (46) is:

$$\sigma = -\int_0^t f_1(t - \tau)\phi(\sigma(\tau))\, d\tau + f_2(t) \tag{48}$$

where $f_1(t)$ is the inverse transform of $1/F_1(s)$ $(f_1(t) = 0, t \leqq 0)$ and $f_2(t)$ is the convolution of $f_1(t)$ and $g_2(t)$. Equation (48) is the integral form of Eqs. (43) that was sought. Note that $f_2(t)$ is a bounded function since all the singularities of both $f_1(t)$ and $g_2(t)$ are in the left-half complex plane.

Next, multiply both sides of Eq. (48) by $\phi(\sigma)$ and integrate the result up to time T to find:

$$\int_0^T \int_0^T f_1(\lambda - \tau)\phi(\sigma(\lambda))\phi(\sigma(\tau))\, d\lambda\, d\tau + \int_0^T \sigma(\lambda)\phi(\sigma(\lambda))\, d\lambda - \int_0^T f_2(\lambda)\phi(\sigma(\lambda))\, d\lambda = 0 \tag{49}$$

Thanks to a theorem due to Bochner (1932) (see Ky Fan (1950)) note that when $1/F_1(s)$ is a positive real function, then the first integral of Eq. (49) is always nonnegative for any function ϕ. In addition, the second integral is always positive since $\sigma\phi(\sigma) > 0$. These two observations about Eq. (49) have the following implications.

The first is that σ is bounded. Indeed, if this were not true, since $f_2(t)$ is bounded, there would exist a time t_0 such that when $T > t_0$, the second integral of Eq. (49) would become larger than the third and then Eq. (49) would require that the sum of two positive numbers be zero. This is clearly absurd and consequently σ must be bounded.

The second is that $\sigma \to 0$ as $T \to \infty$ because then and only then all terms of Eq. (49) would be bounded for all values of T. But $\sigma \to 0$, $T \to \infty$ implies $x \to 0$, $T \to \infty$ (see Eq. (43a)). Therefore, the solutions of Eqs. (43) are asymptotically stable.

In summary, it is found again that the necessary and sufficient condition for the system of Eqs. (43) to be globally asymptotically stable for all members of the class of functions $f(\sigma)$ ($\rho_1 > 0$, $h \geqq 0$) or for the restricted class ($\rho_1 > 0$, $\rho_1 + h > 0$) is that the first approximation be stable for the same functions $f(\sigma)$ respectively.

E. Conditional Global Asymptotic Stability

In Sections IV, C and IV, D it is shown that investigation of the question of global stability of systems describable by Eqs. (43), either by means of the generalized Lur'e method (Eq. (29)) or by means of a constant of the motion (Eq. (49)), leads to conditions that are, in general, more restrictive than required for conditional stability of the first approximation. The purpose of this section is to prove that this overrestriction is not always necessary.

To this end, it is convenient to return to discrete variables and consider the canonical form (Eqs. (13))

$$\begin{aligned} \dot{x}_k &= r_k x_k + \sigma \qquad\qquad k = 1, 2, \cdots m \\ \dot{\sigma} &= \sum_{k=1}^{m} \bar{\beta}_k x_k - (\bar{\rho} + h)\sigma - \phi(\sigma) \end{aligned} \tag{50}$$

There is a class of systems for which when the first approximation is conditionally stable for a specific range of values of h, it is possible to transform the system of Eqs. (50) into another canonical form with respect to the $(m + 1)$ independent variables $x_1, x_2, \ldots x_m, \sigma$ and such that:

$$\dot{y}_i = \lambda_i y_i - \phi(z); \qquad i = 1, 2, \cdots (m+1); \qquad z = \sum_{i=1}^{m+1} k_i y_i \tag{51}$$

where the y_i's are the new variables, the λ_i's are the characteristic values (poles) of the first approximation ($\text{Re}\,\lambda_i < 0$), assumed to be all distinct, and the k_i's are the elements of the $(m + 1)$-column of the inverse of the matrix:

$$[P] = \begin{bmatrix} \lambda_1^m & \lambda_2^m & \cdots & \lambda_{m+1}^m \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_{m+1}^{m-1} \\ \vdots & \vdots & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{52}$$

In other words there is a class of systems for which the first approximation is reduced to a canonical form by means of the Vandermonde matrix P. It can be readily shown that the k_i's satisfy the equation

$$\sum_{i=1}^{m+1} k_i = 1 \tag{53}$$

A procedure similar to the one used in Section IV, C reveals that the system represented by Eqs. (51) is globally asymptotically stable if the function

$$Z(s) = 1 - (-1)^{m+1} \frac{\lambda_1 \lambda_2 \cdots \lambda_{m+1}}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_{m+1})} \tag{54}$$

is positive real without zeros on the $j\omega$-axis or if

$$\text{Re}[1 - Z(j\omega)] < 1 \tag{55}$$

Indeed, when $Z(s)$ is a positive real function it can be immediately verified that the positive definite form

$$V = -\sum_{i=1}^{m+1} \sum_{k=1}^{m+1} \frac{a_i\, a_k}{\lambda_i + \lambda_k} y_i\, y_k + \int_0^z \phi(z)\, dz \tag{56}$$

admits a negative definite total time derivative

$$\dot{V} = -\left[\sum_{i=1}^{m+1} a_i\, y_i - \phi(z)\right]^2 \tag{57}$$

There is a large variety of combinations of the characteristic values λ_i for which $Z(s)$ is positive real. For example, an acceptable combination is when all the λ_i's are in the region defined by the 135° and 225° lines of the complex plane because then each of the factors $\lambda_i/(s - \lambda_i)$ of $(1 - Z(s))$ has a magnitude smaller than unity, for all $s = j\omega$. Other combinations, including λ_i's throughout the left-half complex plane, can

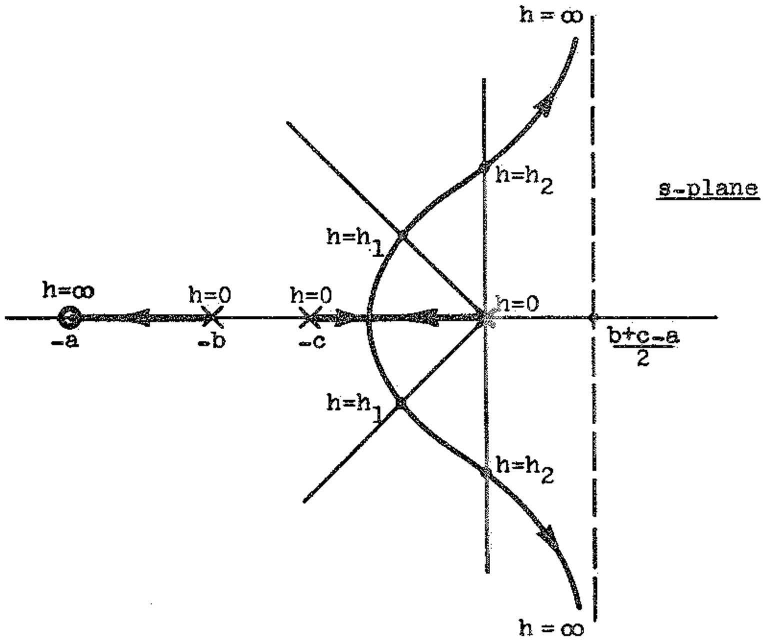

FIG. 2. Locus of the characteristic values of the first approximation of a third order system (Eqs. (7), $n = 2$) with

$$r - G(s) = \frac{s + a}{(s + b)(s + c)}; \qquad b + c - a > 0$$

for $0 < h < \infty$. The system is conditionally stable because for $h > h_2$ the first approximation is unstable and therefore the Lur'e–Letov procedure for global stability is not constructive. For $h < h_1$, however, the characteristic values are in the 135°–225° quadrant and the system is shown to be globally stable by the method described in Section IV, E.

be established through a detailed investigation of the necessary conditions for $Z(s)$ to be positive real.

Such acceptable combinations of characteristic values λ_i could not have been revealed by the procedures presented in Section IV, C. To see this clearly, consider as an example a third order system whose characteristic values as a function of h are given by the root locus of Fig. 2. This is a conditionally stable system and therefore the Liapunov functions proposed in the previous sections are not constructive for the investigation of the global asymptotic stability of the system. On the other hand, if the values of h are restricted so that the characteristic roots lie anywhere between the 135° and 225° line of the complex plane, then according to inequality (55) the system is globally asymptotically stable.

In summary, it is shown that the sufficient conditions for global asymptotic stability of systems represented by the special system of Eqs. (50) are that the first approximation be stable and that the characteristic values satisfy Eq. (55).

A similar procedure is applicable for any system of the form of Eqs. (43). The requirement for global asymptotic stability in that the first approximation be stable and that the function

$$Z_1(s) = 1 + \sum_{i}^{m+1} \frac{k_i \lambda_i}{s - \lambda_i} \tag{54a}$$

be positive real, where the k_i's are defined as previously but the transformation matrix is not the Vandermonde matrix but one which satisfies only the condition that its last row be the vector (1, 1, . . . 1).

V. DISCUSSION

The preceding presentation has revealed several important properties of the nonlinear systems studied by Lur'e as well as the broader class described by Eqs. (19) and (20).

a. It is established quite generally and without reference to specific examples that all systems characterized by the class of nonlinear functions $f(\sigma)$ are globally asymptotically stable, provided that there is only one equilibrium state and the first approximation around this state is unconditionally stable.

b. It is shown that if the first approximation is conditionally stable, then again linear stability guarantees global asymptotic stability provided that the characteristic values are in a certain limited region of the left-half complex plane.

c. The derived conditions for global asymptotic stability can be found either analytically by means of the well known techniques of linear feedback theory or experimentally by means of small perturbation tests.

d. The introduction of the continuum $x(\rho, t)$ expedites the analysis of the problem of stability and eliminates a large variety of over-restrictive stability criteria that have been proposed by Lur'e and Letov.

A question that has not been investigated is "what happens when the first approximation is unstable"? It can be shown that the nonlinear systems under consideration may be Lagrange stable. This topic will be the subject of a future paper.

RECEIVED: January 4, 1963

Note added in proof: After this paper was submitted for publication and while it was being reviewed, Dr. R. E. Kalman published a paper on "Liapunov Functions for the Problem of Lur'e in Automatic Control" in the *Proc. Natl. Acad. Sci. U. S.* **49**, 201–205 (1963). In this publication Kalman derives by means of a different technique some of the results presented in the present communication.

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