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Transfer Functions of Distributed Parameter Nuclear Reactor Systems[1]

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Transfer functions of nuclear reactors and counterflow heat exchangers are derived from the partial differential equations with respect to time and spatial coordinates describing the transient behavior of nuclear power plants. These transfer functions can be approximated by lumped electrical networks and pure delays for analog computer studies. The procedure of approximation is illustrated by specific examples.

INTRODUCTION

The dynamic performance of a nuclear power plant is described by means of partial differential equations with respect to time and spatial variables. Therefore, a nuclear plant is a distributed parameter system. Although this is generally recognized, the transfer functions of nuclear reactors and their associated thermal components are usually derived from lumped parameter models (*1*, *2*, *3*) and thus the partial differential equations reduce to ordinary differential equations.

In this paper an attempt is made to derive the transfer functions of nuclear reactors and counterflow heat exchangers from the partial differential equations with the assumptions of a one-phase reactor coolant and temperature-independent heat transfer coefficients. In the first part of the paper the transfer functions associated with the reactor are derived for two models of heat flow. In the second part the counterflow heat exchanger is studied.

NUCLEAR REACTOR TRANSFER FUNCTIONS

BASIC ASSUMPTIONS

The reactor is heterogeneous with fuel, moderator, and one-phase coolant regions in which heat exchange is mostly one-dimensional from the fuel region perpendicularly to the coolant through the moderator. The reactor is considered as made of groups of three slabs (Fig. 1) without any loss of generality because such a geometry can be adapted to many reactor designs (*4*, *5*).

The reactor transfer functions are derived under the following assumptions:

a. The heat is released in the fuel slabs and is carried away by the nearest coolant slabs.

b. The heat sources are independent of spatial coordinates. The general case of space-dependent heat sources has been considered elsewhere by one of the authors (*6*).

c. The heat is transferred by conduction in the fuel and moderator regions and by convection in the coolant region.

d. The temperature across the coolant region is constant because of turbulence of the fluid.

e. The cross section of the coolant slabs is varied in such a way that the outlet coolant temperature is equal for any slab. Consequently the temperature of the moderator is approximately constant in any vertical plane perpendicular to the coolant slabs.

f. The inputs to the reactor are coolant velocity and coolant inlet temperature. The outputs are coolant outlet temperature and heat released in the fuel.

Under these assumptions the cell shown in Fig. 2 is representative of the entire reactor.

TRANSFER FUNCTIONS

The nomenclature of the following sections is:

c_1, c_2, c': specific heats of the moderator, fuel,

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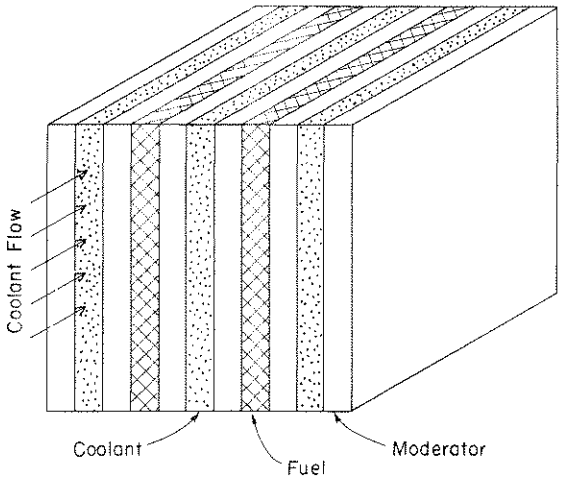

FIG. 1. Equivalent reactor.

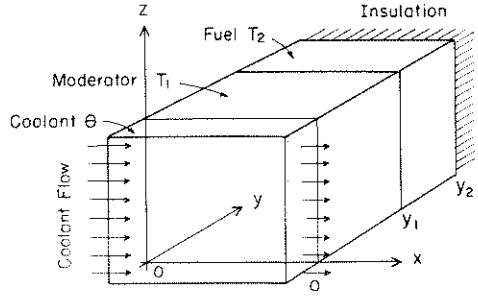

FIG. 2. Approximate reactor core subcell; Model I.

and coolant, respectively, in BTU/lb °F

h: heat transfer coefficient at the coolant-moderator interface in BTU/ft^2 hr °F

k, k_1, k_2: conductivities of the fictitious solid (Model II), the moderator, and the fuel, respectively, in BTU/ft hr °F

k_{eff}: effective multiplication factor

l: average length of the coolant slabs in the core in ft

l^*: effective neutron lifetime in hr

n: neutron density in neutrons/ft^3

q: $\sqrt{s/\alpha}$ in ft^{-1}

q_1: $\sqrt{s/\alpha_1}$ in ft^{-1}

q_2: $\sqrt{s/\alpha_2}$ in ft^{-1}

s: complex frequency in hr^{-1}

v: coolant velocity in ft/hr

A^*: effective surface of heat exchange between moderator and coolant in ft^2

C_i: ith delayed neutron precursor density in atoms/ft^3

M': mass of the coolant in reactor in lb

Q, Q_0: heat source density in the fuel and in the fuel at operating temperature, respectively, in BTU/ft^3 hr

S: $1/kA^*$ heat flow in °F/ft

T, T_1, T_2: temperatures of the fictitious solid (Model II), the moderator, and the fuel, respectively, in °F

T_m, T_f: Average temperatures of the moderator and fuel, respectively, in °F

α, α_1, α_2: thermal diffusivities of the fictitious solid, (Model II), the moderator, and the fuel in ft^2/hr

β: total fraction of delayed neutrons

β_i: fraction of delayed neutrons of the ith species

θ, θ_0, θ_l: coolant temperature, inlet temperature, and outlet temperature, respectively, in °F

λ_i: precursor decay constant (ith species) in hr^{-1}

ρ, ρ_1, ρ_2: densities of the fictitious solid (Model II), the moderator, and the fuel, respectively, in lb/ft^3

ρ^*: reactivity

Any overbarred symbol implies the Laplace transform of the increment above the steady-state value of the quantity in question.

The transfer functions to be derived are depicted schematically in Fig. 3 and are self-evident. The computations are simple but rather long; hence the derivation of only one transfer function will be given in full detail.

Temperature and Power Transfer Functions—Model I

The partial differential equations for the fuel, moderator, and coolant temperatures of the representative cell (Fig. 2) are:

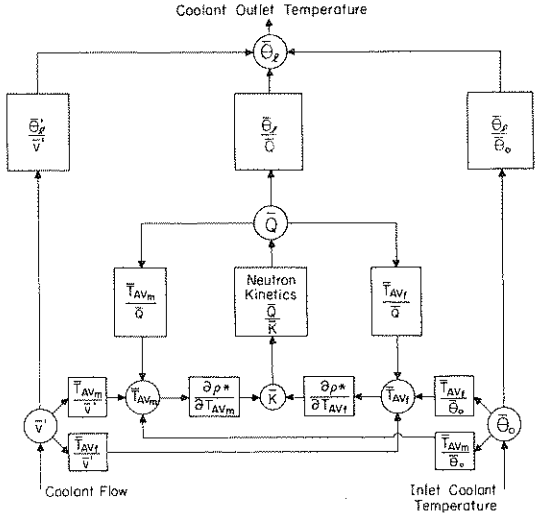

FIG. 3. Reactor block diagram.

$$k_2 \frac{\partial^2 T_2}{\partial y^2} + Q = \rho_2 c_2 \frac{\partial T_2}{\partial t} \tag{1}$$

$$k_1 \frac{\partial^2 T_1}{\partial y^2} = \rho_1 c_1 \frac{\partial T_1}{\partial t} \tag{2}$$

$$v \frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial t} = b(T_1 - \theta) \tag{3}$$

where time t is in hours and distances x and y are in feet, and b is $hA^*/M'c'$ in hr^{-1}. The boundary conditions are:

$$y = y_2 \qquad \frac{\partial T_2}{\partial y} = 0 \tag{4}$$

$$y = y_1 \qquad k_2 \frac{\partial T_2}{\partial y} = k_1 \frac{\partial T_1}{\partial y} \tag{5}$$

$$T_2 = T_1 \tag{6}$$

$$y = 0 \qquad k_1 \frac{\partial T_1}{\partial y} = h(T_1 - \theta) \tag{7}$$

$$x = 0 \qquad \theta = \theta_0(t) \tag{8}$$

The system of partial differential equations (1) to (3) is transformed into a system of ordinary differential equations by means of the Laplace transform. Then the coolant inlet temperature—outlet temperature transfer function is found by solving the latter system and is:

$$\bar{\theta}_l/\bar{\theta}_0 = \exp(-\Lambda l/\Pi) \tag{9}$$

with

$$\begin{aligned}\Lambda = {} & s \frac{q_2 k_2}{q_1 k_1} \sinh q_1 y_1 \sinh q_2(y_2 - y_1) \\ & + (s + b) \frac{q_1 k_1}{h} \sinh q_1 y_1 \cosh q_2(y_2 - y_1) \\ & + s \cosh q_1 y_1 \cosh q_2(y_2 - y_1) \\ & + (s + b) \frac{q_2 k_2}{h} \cosh q_1 y_1 \sinh q_2(y_2 - y_1)\end{aligned} \tag{10}$$

$$\begin{aligned}\Pi = {} & v \frac{q_2 k_2}{q_1 k_1} \sinh q_1 y_1 \sinh q_2(y_2 - y_1) \\ & + v \frac{q_1 k_1}{h} \sinh q_1 y_1 \cosh q_2(y_2 - y_1) \\ & + v \cosh q_1 y_1 \cosh q_2(y_2 - y_1) \\ & + v \frac{q_2 k_2}{h} \cosh q_1 y_1 \sinh q_2(y_2 - y_1)\end{aligned} \tag{11}$$

The intermediate steps of the derivation are omitted here but are given in the next section for a similar problem.

The power-outlet temperature transfer function is:

$$\frac{\bar{\theta}_l}{\bar{Q}} = \frac{b}{h q_2} \frac{1 - e^{(\Lambda/\Pi)l}}{\Lambda} \sinh q_2(y_2 - y_1) \tag{12}$$

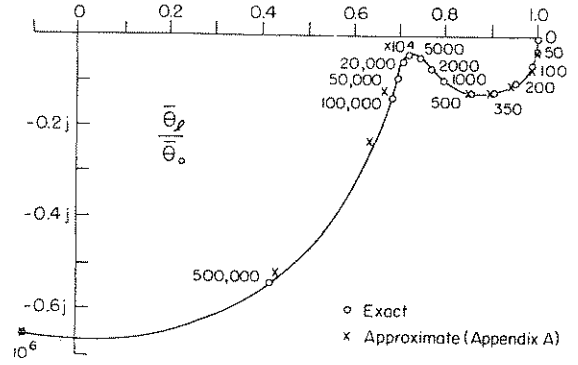

FIG. 4. Model I; temperature transfer function for $s = j\omega$.

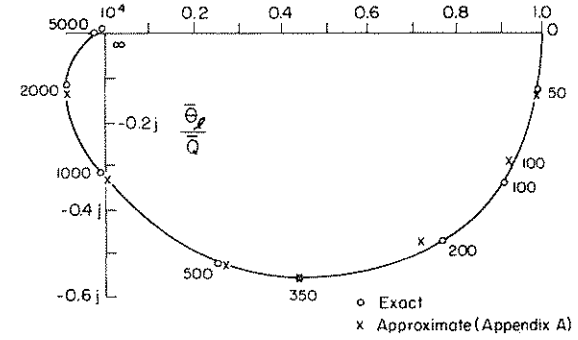

FIG. 5. Model I; power transfer function for $s = j\omega$.

Both transfer functions (9) and (12) have been plotted for all values of s along the j-axis in Figs. 4 and 5, respectively, for values of the parameters given in Table I. However, they are rather cumbersome to use and, therefore, it is appropriate to examine whether it is possible to replace the present model of heat transfer by another which might lead to simpler mathematical expressions.

Various possibilities have been studied (*6*) but only the one described subsequently gives results comparable to the results of Model I.

Temperature and Power Transfer Functions—Model II

A representative cell of the approximate reactor model is given in Fig. 6. It consists of a slab of a fictitious solid whose thermodynamic properties are an appropriate average of the moderator and fuel properties of the cell given in Fig. 2 to be subsequently defined. The distributed heat source is replaced by an equivalent heat flow at the fuel-moderator interface. The other side of the fictitious solid is cooled by the coolant flow.

The heat balance equations for the solid and coolant are:

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{13}$$

TABLE I

DATA FOR THE COMPUTATION OF THE TEMPERATURE AND POWER TRANSFER FUNCTIONS

b	$= hA/(M'c')$	$= 226{,}000\ \text{hr}^{-1}$
h	= heat transfer coefficient	$= 1{,}100\ \text{BTU/ft}^2\ \text{hr}\ ^\circ\text{F}$
k_1	= conductivity of moderator	$= 36.2\ \text{BTU/hr ft}\ ^\circ\text{F}$
l	= average length of the coolant slabs	$= 3\ \text{ft}$
v	= coolant velocity in the slabs	$= 1{,}710{,}000\ \text{ft/hr}$
y_0	= weighted moderator thickness	$= 2.217 \times 10^{-2}\ \text{ft}$
y_1	= moderator thickness	$= 2.83 \times 10^{-2}\ \text{ft}$
$y_2 - y_1$	= fuel thickness	$= 1.785 \times 10^{-2}\ \text{ft}$
α	= average thermal diffusivity	$= 0.45\ \text{ft}^2/\text{hr}$

$$v\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial t} = b(T - \theta) \tag{14}$$

The boundary conditions are:

$$y = y_0 \qquad S = \frac{Q(y_2 - y_1)}{k} = \frac{\partial T}{\partial y} \tag{15}$$

$$y = 0 \qquad k\frac{\partial T}{\partial y} = h(T - \theta) \tag{16}$$

$$x = 0 \qquad \theta = \theta_0(t) \tag{17}$$

The Laplace transforms of Eqs. (13) to (15) and (17) are:

$$\frac{d^2\overline{T}}{dy^2} = \frac{s}{\alpha}\overline{T} \tag{18}$$

$$v\frac{d\bar{\theta}}{dx} + s\theta = b(\overline{T} - \bar{\theta}) \tag{19}$$

$$y = y_0 \qquad \bar{S} = \frac{d\overline{T}}{dy} \tag{20}$$

$$x = 0 \qquad \bar{\theta} = \bar{\theta}_0 \tag{21}$$

The solution of Eq. (18) is:

$$\overline{T} = A\cosh qy + B\sinh qy \tag{22}$$

Equation (22) is used in Eq. (16), which reduces to:

$$\bar{\theta} = A - (kBq/h) \tag{23}$$

The use of Eqs. (22) and (23) in Eqs. (19) and (20) leads to:

$$v\frac{dA}{dx} - \frac{kq}{h}v\frac{dB}{dx} + sA - s\frac{kqB}{h} = b\frac{k}{h}Bq \tag{24}$$

$$\bar{S} = qA\sinh qy_0 + qB\cosh qy_0 \tag{25}$$

The elimination of B yields:

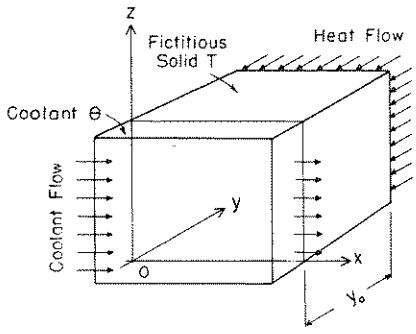

FIG. 6. Fictitious reactor core subcell; Model II.

$$\frac{dA}{dx} + f_1 A = f_2\bar{S} \tag{26}$$

where

$$f_1 = \frac{s}{v} + \frac{b}{v}\cdot\frac{(kq/h)\tanh qy_0}{1 + (kq/h)\tanh qy_0} \tag{27}$$

$$f_2 = \frac{k(s+b)}{vh\cosh qy_0}\cdot\frac{1}{1 + (kq/h)\tanh qy_0} \tag{28}$$

The solution of Eq. (26) is:

$$A = e^{-f_1 x}\left[G + \bar{S}f_2\frac{e^{+f_1 x} - 1}{f_1}\right] \tag{29}$$

where G is a constant which will be determined. The elimination of B between Eqs. (23) and (25) leads to an equation for $\bar{\theta}$ in which substitution of A from Eq. (29) yields the coolant temperature:

$$\bar{\theta} = \bar{\theta}_0 e^{-f_1 x} + \bar{S}\frac{k}{h\cosh qy_0}\left(\frac{s+b}{vf_1} - 1\right)(1 - e^{-f_1 x}) \tag{30}$$

where G has been chosen such that $\bar{\theta} = \bar{\theta}_0$ at $x = 0$.

Consequently the inlet temperature–outlet temperature transfer function is:

$$\bar{\theta}_l/\bar{\theta}_0 = \exp(-f_1 l) \tag{31}$$

and the power-outlet temperature transfer function is:

$$\frac{\bar{\theta}_l}{\bar{Q}} = \frac{y_2 - y_1}{h\cosh qy_0}\left(\frac{s+b}{vf_1} - 1\right)(1 - e^{-f_1 l}) \tag{32}$$

The parameters of the fictitious solid are derived from a comparison of the asymptotic behavior of the transfer functions of Models I and II along the j-axis.

(*1*) *Temperature transfer functions*:

For low frequencies

$$\text{Model I: } \frac{\bar{\theta}_l}{\bar{\theta}_0} = \exp\left\{(-j\omega l/v)\left[1 + b\left(\frac{k_2}{\alpha_2}\frac{y_2 - y_1}{h} + \frac{k_1 y_1}{\alpha_1 h}\right)\right]\right\} \simeq 1 \tag{33}$$

$$\text{Model II: } \frac{\bar{\theta}_l}{\bar{\theta}_0} = \exp\left\{-(j\omega l/v)\left[1 + b\,\frac{k}{h}\frac{y_0}{\alpha}\right]\right\} \simeq 1 \tag{34}$$

Thus, in order to have good agreement at low frequencies, the parameters of Model II are taken such that

$$\frac{k y_0}{\alpha} = \frac{k_1 y_1}{\alpha_1} + \frac{k_2(y_2 - y_1)}{\alpha_2} \tag{35}$$

with the additional conditions

$$k = k_1 \tag{36}$$

$$\frac{1}{\alpha} = \left(\frac{\rho c}{k}\right)_{av} = \frac{M_1(1/\alpha_1) + M_2(1/\alpha_2)}{M_1 + M_2} \tag{37}$$

where M_1 and M_2 are the mass of moderator and mass of fuel, respectively.

For high frequencies

$$\text{Models I and II:} \qquad \bar{\theta}_l/\bar{\theta}_0 = e^{-bl/v}e^{-j\omega l/v} \tag{38}$$

(2) *Power transfer functions*:

For low frequencies

$$\text{Models I and II: } \bar{\theta}_l/\bar{Q} = (y_2 - y_1)(bl/(vh)) \tag{39}$$

For high frequencies

$$\text{Models I and II:} \qquad \bar{\theta}_l/\bar{Q} = 0 \tag{40}$$

Both transfer functions (31) and (32) have been plotted for $s = j\omega$ in Figs. 7 and 8, respectively, for values of the parameters given in Table I. The comparison of these figures to Figs. 4 and 5 shows that Model II is a good approximation of the original reactor and since this model leads to simpler mathematical expressions, it will be adopted for further calculations, unless otherwise stated.

Coolant Velocity Transfer Function

The transfer function between coolant velocity and outlet coolant temperature is derived by means of a perturbation method with the assumption of small variations of coolant velocity and velocity-dependent heat transfer coefficients.

The steady-state solution of Eqs. (13) and (14) is

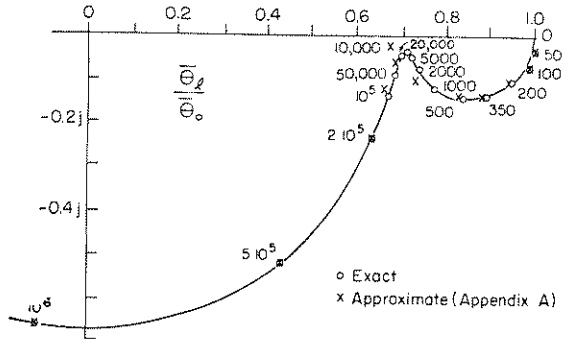

FIG. 7. Model II; temperature transfer function for $s = j\omega$.

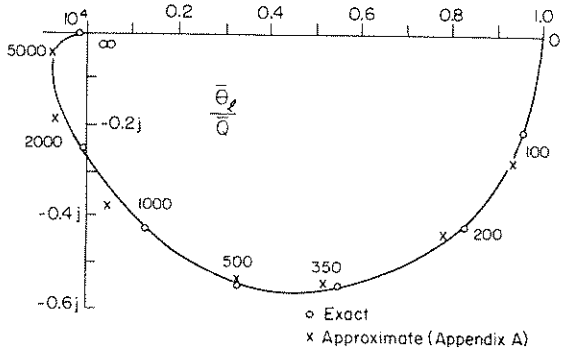

FIG. 8. Model II; power transfer function for $s = j\omega$.

$$T = \theta_0 + \frac{kS_0}{h}\left(1 + \frac{bx}{v}\right) + S_0 y \tag{41}$$

$$\theta = \theta_0 + \frac{bkS_0 x}{vh} \tag{42}$$

$$T - \theta = kS_0/h \quad \text{at} \quad y = 0 \tag{43}$$

If a "prime" indicates the variation of a physical quantity for a variation of velocity, the equations for the moderator and coolant temperatures are

$$\frac{\partial^2 T'}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T'}{\partial t} \tag{44}$$

$$v\frac{\partial \theta'}{\partial x} + v'\frac{\partial \theta}{\partial x} + \frac{\partial \theta'}{\partial t} = b(T' - \theta') + b'(T - \theta) \tag{45}$$

with the boundary conditions

$$y = y_0 \qquad \frac{\partial T'}{\partial y} = 0 \tag{46}$$

$$y = 0 \qquad k\frac{\partial T'}{\partial y} = h(T' - \theta') + h'(T - \theta) \tag{47}$$

$$x = 0 \qquad \theta_0' = 0 \tag{48}$$

After replacement of θ and T by their steady-state values, the system of partial differential equations

(44) and (45) is transformed into a system of ordinary differential equations by means of the Laplace transform. The solution of the latter system leads to the coolant velocity-outlet coolant temperature transfer function:

$$\frac{\bar{\theta}_l'}{\bar{v}'} = \frac{kS_0}{v^2hf_1}\{0.8(s - vf_1) + b\}\{1 - e^{-f_1l}\} \quad (49)$$

In the derivation, it is assumed that the parameters b and h vary like the 0.8 power of the coolant velocity.

Temperature Dependence of Reactivity

The reactivity depends on the average moderator and fuel temperatures. Any variation of coolant inlet temperature, heat rate released in the fuel, or coolant velocity affects the reactor temperatures.

The variation of reactivity is given by:

$$\delta\rho^* = \left(\frac{\partial\rho^*}{\partial\bar{T}_m}\right)\delta T_m + \left(\frac{\partial\rho^*}{\partial T_f}\right)\delta T_f \quad (50)$$

The partial derivatives of reactivity with respect to the average temperatures T_m and T_f can be easily found (7).

The average temperatures are derived by integrating the temperatures for one cell and weighing the average cell temperature by the square of the neutron flux in a first-order approximation, and further integrating throughout the core. However, in the present case, the neutron density and heat rate have been assumed constant and therefore the average temperature of one cell is adequate. Consequently

$$(T_i)_{av} = \frac{1}{l(y_a - y_b)}\int_{y_a}^{y_b}\int_0^l T_i(x,y)\,dx\,dy \quad (51)$$

Moderator Average Temperature and Transfer Functions

The temperature of the fictitious solid is a good approximation to the moderator temperature and it will be used to evaluate the average temperature of the latter.

The temperature of the fictitious solid is a solution of Eqs. (13) and (14) and is $A'\cosh qy + B'\sinh qy$, where A' and B' are constants determined from the boundary conditions (15) to (17). After some algebra and use of a perturbation calculation, it is found that the average temperature of the moderator is dependent on coolant inlet temperature, power, and coolant velocity, and is given by

$$\begin{aligned}\bar{T}_m = {} & \frac{\tanh qy_0}{f_1lqy_0\left(1 + \frac{kq}{h}\tanh qy_0\right)}(1 - e^{-f_1l})\bar{\theta}_0 \\ & + \left\{\frac{\tanh qy_0}{hqy_0\left(1 + \frac{kq}{h}\tanh qy_0\right)\cosh qy_0}\right. \\ & \times\left[\frac{1 - e^{-f_1l}}{f_1l} + \frac{s + b}{vf_1}\left(1 - \frac{1 - e^{-f_1l}}{f_1l}\right)\right] \\ & \left. + \frac{\cosh qy_0 - 1}{kq^2y_0\cosh qy_0}\right\}(y_2 - y_1)\bar{Q} \quad (52) \\ & + S_0\frac{k\tanh qy_0}{hqy_0v} \\ & \times\left(\frac{1 - e^{-f_1l}}{f_1l}\right)\left(0.8 - \frac{0.8\,s + b}{vf_1}\right) \\ & + \frac{0.8\,s + b}{vf_1} \\ & \frac{}{1 + kq/h\tanh qy_0}\bar{v}'\end{aligned}$$

Fuel Average Temperature and Transfer Functions

The computations for the fuel average temperature are similar to the ones for the moderator. However, in this case, Model I is used, since Model II does not take the fuel volume properly into account.

The solution of the system of Eqs. (1) to (3) with the proper boundary conditions yeilds

$$\begin{aligned}\bar{T}_f = {} & \frac{v}{l(y_2 - y_1)}\left[\frac{1 - e^{-\Lambda/\Pi}}{q_2\Lambda}\sinh q_2(y_2 - y_1)\right]\bar{\theta}_0 \\ & + \left[\frac{vb}{lq_2^2(y_2 - y_1)}\,\frac{(l(\Lambda/\Pi) - 1 + e^{-(\Lambda/\Pi)l})}{\Lambda^2h}\right. \\ & \left.\times\sinh^2 q_2(y_2 - y_1) + \frac{\alpha_2}{k_2s}\right] \\ & \bar{Q} + \frac{\bar{Q}_0\sinh q_2(y_2 - y_1)}{hq_2} \quad (53) \\ & \times\left[0.8\,\frac{1 - e^{-(\Lambda/\Pi)l}}{\Lambda l}\right. \\ & \left. + \frac{0.8\,s + b}{v\Lambda}\left(1 - \frac{1 - e^{-(\Lambda/\Pi)l}}{l\Lambda}\,\Pi\right)\right]\bar{v}'\end{aligned}$$

Neutron Kinetics Transfer Function

The power released in a nuclear reactor is proportional to the neutron flux which can be computed from the neutron kinetics equations:

$$\frac{dn}{dt} = k_{\text{eff}} \frac{k_{\text{eff}}(1-\beta)-1}{l^*} n + \sum_i \lambda_i C_i \quad (54)$$

$$\frac{dC_i}{dt} = \beta_i \frac{k_{\text{eff}}}{l^*} n - \lambda_i C_i \quad (55)$$

For small variations of k_{eff} around the steady-state value the neutron kinetics equations can be linearized, and the following transfer function is found (7):

$$\frac{\bar{Q}}{\bar{k}} = Q_0 \frac{1 - s \sum_i \frac{\beta_i}{s+\lambda_i}}{s\left(l^* + \sum_i \frac{\beta_i}{s+\lambda_i}\right)} \quad (56)$$

Thus, the derivation of all the transfer functions of the reactor block diagram of Fig. 3 is complete. In case of an analog simulation of the reactor the previously derived transfer functions can be approximated by lumped networks and delay lines. The procedure to be used is briefly illustrated in the Appendix by means of a numerical example.

COUNTERFLOW HEAT EXCHANGER TRANSFER FUNCTIONS

Heat Exchanger Description

The heat exchanger is of the counterflow type and consists of an economizer, a boiler, and a superheater. The primary loop is fed with a one-phase fluid; the secondary loop is fed with water (Fig. 9).

Each of the three components of the heat exchanger may be visualized as consisting of a single tube containing the coolant in the proper phase and surrounded by the hot fluid, which in turn is contained in a concentric shell (Fig. 10).

Transfer Functions of the Heat Exchanger

The transfer functions between the incoming fluid temperatures and the outgoing fluid temperatures will be derived for the superheater and the boiler. The transfer functions of the economizer are similar to the ones of the superheater since in both cases it is assumed that the secondary coolant is in the liquid or vapor phase, respectively.

The assumptions behind the derivation of the transfer functions are:

(a) The turbulence is very high in both fluids.

(b) The heat conduction along the axial direction of the tube and the shell is zero.

(c) The heat conductivity of the tube and the shell along the radius is infinite.

(d) The heat losses to the surroundings are zero.

(e) The heat transfer coefficients are constant.

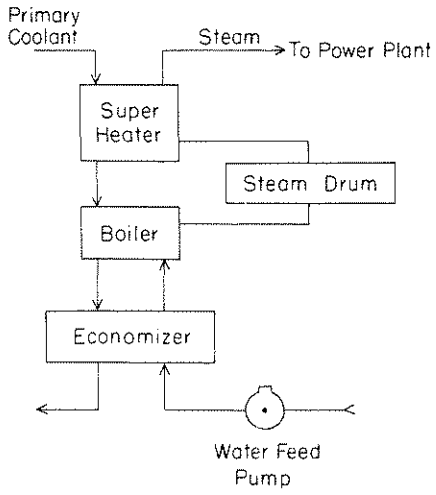

Fig. 9. Heat exchanger diagram.

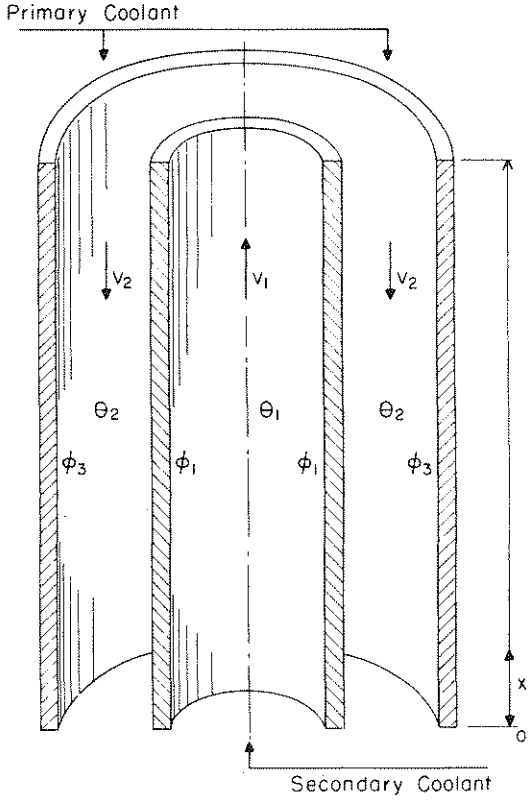

Fig. 10. Heat exchanger schematic view.

The nomenclature of the following sections is:

$a_1 = (h_1A_1)/(M_1c_1)$; $a_2 = (h_2A_2)/(M_2c_2)$; etc. in hr^{-1}

$b_1 = (h_1A_1)/(M_3c_3)$; $b_2 = (h_2A_2)/(M_2c_2)$; $b_3 = (h_3A_3)/(M_4c_4)$ in hr^{-1}

c_1, c_2, c_3, c_4: specific heats in BTU/lb °F of water, hot fluid, tube, and shell, respectively

h_1, h_2, h_3: heat transfer coefficients in BTU/ft^2 hr °F on the water side, the hot fluid side, and the shell side, respectively.

l, l', l'': lengths in ft of superheater, boiler, and economizer, respectively

v_1, v_2: water and hot fluid velocities in ft/hr, respectively

x: running length in ft along the heat exchanger

A_1, A_2, A_3: surface areas in ft^2 of the tube on the water side, of the tube on the hot fluid side, and of the shell, respectively

F: fluid flow in ft^3/hr through mixing volume

M_1, M_2, M_3, M_4: masses in lb of water inside the tubes, of hot fluid inside the heat exchanger, of the tube, and of the shell, respectively

V: mixing volume in ft^3

θ_1, θ_2: water and hot fluid temperatures, °F, respectively

θ_{10}, θ_{1l}: inlet and outlet water temperatures, in °F, respectively

θ_{20}, θ_{2l}: outlet and inlet hot fluid temperatures, in °F, respectively

φ_1, φ_3: tube and shell temperatures, in °F, respectively

$$f_1 = (1/v_1)\{s + a_1[(s + b_2)/(s + b_1 + b_2)]\}$$
$$f_2 = (1/v_2)\{s + a_2[(s + b_1)/(s + b_1 + b_2)] + a_3[s/(s + b_3)]\}$$
$$f_3 = (1/v_1)[a_1b_2/(s + b_1 + b_2)]$$
$$f_4 = (1/v_2)[a_2b_1/(s + b_1 + b_2)]$$
$$f_5 = s + a_2[(s + b_1)/(s + b_1 + b_2)] + a_3[s/(s + b_3)]$$
$$\alpha = (f_2 - f_1)/2$$
$$\beta = \tfrac{1}{2}[(f_1 + f_2)^2 - 4f_3f_4]^{1/2}$$

The α, β, f_1, $\cdots$, f_5 are all in ft^{-1}. Any overbarred symbol implies the Laplace transform of the increment above the steady-state value of the quantity in question.

Superheater Transfer Functions

A heat balance in an elementary volume of the superheater results in the following equations for the steam, the internal tube, the hot fluid, and the external shell, respectively (Fig. 10):

$$\frac{\partial\theta_1}{\partial t} + v_1\frac{\partial\theta_1}{\partial x} = a_1(\varphi_1 - \theta_1) \tag{57}$$

$$\frac{\partial\theta_1}{\partial t} = b_1(\theta_1 - \varphi_1) + b_2(\theta_2 - \varphi_1) \tag{58}$$

$$\frac{\partial\theta_2}{\partial t} - v_2\frac{\partial\theta_2}{\partial x} = a_2(\varphi_1 - \theta_2) + a_3(\varphi_3 - \theta_2) \tag{59}$$

$$\frac{\partial\varphi_3}{\partial t} = b_3(\theta_2 - \varphi_3) \tag{60}$$

The boundary conditions are:

$$\begin{aligned} x &= 0 \qquad \theta_1 = \theta_{1l'} \\ x &= l \qquad \theta_2 = \theta_{2l} \end{aligned} \tag{61}$$

The solution of the system in Eqs. (57) to (60) by means of the Laplace transform leads to:

$$\bar\theta_1 = \bar\theta_{1l'}\frac{e^{\alpha x}\sinh[\beta(l - x) + \lambda]}{\sinh(\beta l + \lambda)} + \bar\theta_{2l}\sqrt{\frac{f_3}{f_4}}\frac{e^{\alpha(x-l)}\sinh\beta x}{\sinh(\beta l + \lambda)} \tag{62}$$

$$\bar\theta_2 = \bar\theta_{1l'}\sqrt{\frac{f_4}{f_3}}\frac{e^{\alpha x}\sinh\beta(l - x)}{\sinh(\beta l + \lambda)} + \bar\theta_{2l}\,e^{\alpha(x-l)}\frac{\sinh(\beta x + \lambda)}{\sinh(\beta l + \lambda)} \tag{63}$$

where λ is defined by the equation:

$$\sinh\lambda = \beta/\sqrt{f_3f_4} \tag{64}$$

The transfer functions relating hot fluid and steam inputs to hot fluid and steam outputs are obtained by replacing x by 0 or l in accordance with the case. They are given explicitly in Fig. 11.

Boiler Transfer Function

In the boiler section of the heat exchanger the secondary coolant is in the saturated vapor phase and its temperature is constant when the steam pressure is constant. The heat balance equations for the internal tube, the hot fluid, and the shell are the same as Eqs. (58) to (60) with θ_1 constant.

The boundary conditions are:

$$x = 0, \qquad \theta_2 = \theta_{2l'} \tag{65}$$

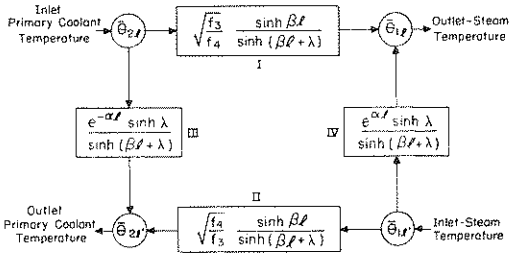

Fig. 11. Superheater transfer functions.

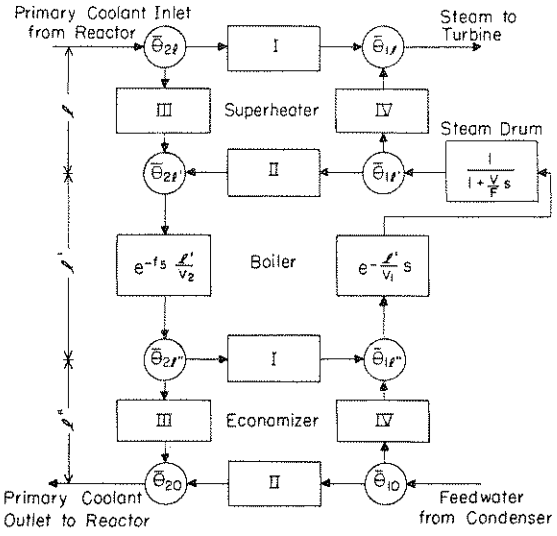

FIG. 12. Heat exchanger block diagram.

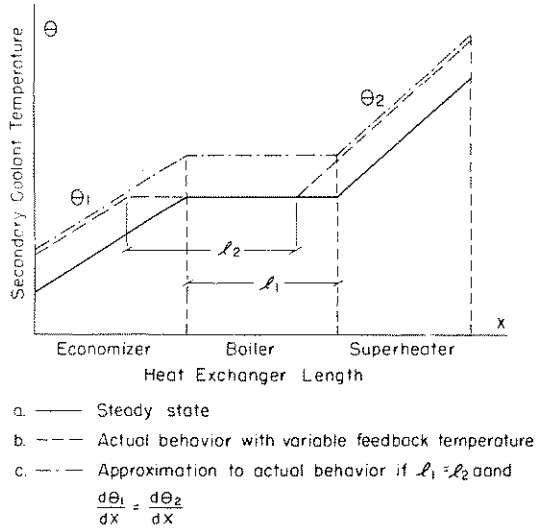

FIG. 13. Secondary coolant temperature across heat exchanger.

The solution of this system of equations in terms of the complex frequency gives the transfer function as:

$$\bar{\theta}_{2l''} = \bar{\theta}_{2l'} \exp(-f_s l'/v_2) \qquad (66)$$

FLUID FLOW TRANSFER FUNCTION

The effect of flow variations of the fluids going through the heat exchanger has been studied elsewhere (*6*) by a perturbation method but the formulae obtained are complicated and long and they are omitted in this paper.

HEAT EXCHANGER BLOCK DIAGRAM

The time behavior of the heat exchanger under inlet temperature variations is a problem of moving boundary conditions. For instance, as the inlet water temperature increases the economizer tube length is decreased, boiling starts earlier, and hence the superheater tube length is increased.

A first approximation to this can be obtained by considering the lengths of the components of the heat exchanger constant and by assembling in a block diagram (Fig. 12) the transfer functions of the superheater, the boiler, the steam drum, and the functions of the economizer which are similar to the transfer functions of the superheater. As a perturbation in the vapor inlet temperature takes time to propagate through the boiler, a pure delay transfer function has also been included. The spproximation is good in a quasi-steady state when the gradient of the secondary coolant temperature in the economizer and the superheater are equal and the boiling tube length is constant (Fig. 13*a*, *b*, *c*).

CONCLUSION

The transfer functions of the main components of a nuclear power plant, i.e., the reactor and the heat exchanger have been derived. Those components are distributed parameter systems and the transfer functions are obtained by integration of a set of partial differential equations. The method is simple and straightforward when the Laplace transform technique is used.

The transfer functions obtained can be approximated for low frequencies by rational functions of the complex frequency s and exponentials of s. Thus they can be represented by electric networks and delay lines.

As a final word, it is hoped that this work will be useful for the analog study of reactor systems as it gives a better representation of the reactor and its associated steam plant.

APPENDIX: LUMPED NETWORK AND TIME LAG APPROXIMATIONS TO TRANSFER FUNCTIONS

As the developed transfer functions may be used for an analog study of the reactor system, they are hereafter approximated by electric lumped networks and delay lines. The equivalent networks are derived on the basis of the following requirements:

They give the same behavior at zero and infinite frequencies when compared to the exact transfer functions.

They fit the exact transfer functions on a frequency range where the attenuation varies from 0 to 20 db.

Since the distributed parameter systems are not phase minimum, the standard methods of approxi-

mation in the frequency domain do not apply. In this case the type of approximate transfer function is chosen by inspection of the $s = j\omega$ plot of the approximated transfer function. The parameters of the approximate transfer function are adjusted by trial and error.

The approximate temperature and power transfer functions, for instance, for models I and II are:

$$\text{Models I and II } \frac{\bar{\theta}_l}{\bar{\theta}_0} = e^{-bl/v} + \frac{\left(1 - e^{-bl/v}\right)\left(1 + \frac{s}{782}\right)}{\left(1 + \frac{s}{2500}\right)\left(1 + \frac{s}{312}\right)} \cdot e^{-sl/v} \quad \text{(A1)}$$

(Figs. 4 and 7)

$$\text{Model I } \frac{\bar{\theta}_l}{\bar{Q}} \propto \frac{1}{1 + \frac{s}{355}} \cdot \frac{1 - \frac{s}{5750}}{1 + \frac{s}{5750}} \quad \text{(Fig. 5)} \quad \text{(A2)}$$

$$\text{Model II } \frac{\theta_l}{\bar{Q}} \propto \frac{1}{1 + \frac{s}{405}} \cdot \frac{1 - \frac{s}{7650}}{1 + \frac{s}{7650}} \quad \text{(Fig. 8)} \quad \text{(A3)}$$

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