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Theoretical and Experimental Criteria for Nonlinear Reactor Stability

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Criteria for nonlinear asymptotic stability of nuclear reactor systems representable by point reactor kinetics are derived. The reactor system model includes all delayed neutron precursors and linear feedback. The results are also extended to systems where the feedback is not linear.

A procedure is outlined for the interpretation of the criteria by means of small-amplitude perturbation tests. The procedure is illustrated by using experimental transfer function measurements from Experimental Boiling Water Reactor.

I. INTRODUCTION

The purpose of this paper is to present new criteria for nonlinear stability of nuclear reactor systems representable by point reactor kinetics, including all delayed neutron precursors and linear or nonlinear feedback.

It has been shown that stability analyses of nonlinear point reactor kinetics equations, resulting in criteria that do not include the delayed neutron precursor constants, are either over-restrictive or nonconservative. For example, when the feedback transfer function is a lagging function of real frequencies, as for instance in the Experimental Boiling Water Reactor (EBWR), the analysis is over-restrictive because delayed neutrons may relax substantially the design requirements for stability[1]. On the other hand, when the feedback transfer function is a leading function of real frequencies, the analysis may be nonconservative, because the reactor model without delayed neutrons may be stable at all operating power levels, while the model with delayed neutrons may not be stable[2-4].

In addition, it has been recognized that criteria which guarantee absolute asymptotic stability, namely asymptotic stability over an infinite range of operating power levels, may be over-restricting the reactor design and be impractical[1]. The impracticality arises from the fact that no real reactor can be operated at very high power levels because of heat-transfer limitations.

These remarks suggest that physically meaningful stability requirements should include the delayed neutron precursor constants and a finite range of operating power levels. In what follows, new stability criteria are presented that do incorporate these suggestions. Also, a systematic procedure is outlined for the interpretation of the criteria in terms of small-amplitude reactivity transfer function experiments. The criteria are quite general in the sense that they are less restrictive than, and they incorporate as special cases, all other similar criteria derived to date.

II. REACTOR MODEL

First, it is assumed that the reactor has a unique equilibrium operating power level P_1 for a given constant reactivity input at $t = 0$ and that the

[1]E. P. GYFTOPOULOS, *General Reactor Kinetics*, Vol. I., Chap. 3, "The Technology of Nuclear Reactor Safety," 175-204, edited by T. J. Thompson and J. G. Beckerley, MIT Press (1965).

[2]J. CHERNICK, "A Review of Nonlinear Reactor Dynamics Problems," BNL-774, Brookhaven National Laboratory, Upton, N. Y. (1962).

[3]H. B. SMETS, "On the Effect of Delayed Neutrons in Reactor Dynamics" (to be published).

[4]W. BARAN and K. MEYER, "Effect of Delayed Neutrons on the Stability of a Nuclear Power Reactor" *Nucl. Sci. Eng.*, 24, 356-361 (1966).

feedback effects and external controls are linearly related to the power level. Nonlinear feedback is considered later. Thus the kinetics equations for $t > 0$ and in terms of normalized, dimensionless, and incremental variables can be written in the form:

$$\frac{dp(t)}{dt} = -\sum_i^m \frac{\beta_i}{\Lambda}\left[p(t)-c_i(t)\right] + k(t), \tag{1}$$

$$\frac{dc_i(t)}{dt} = \lambda_i\left[p(t)-c_i(t)\right]; \qquad i = 1,2,\ldots m, \tag{2}$$

$$k(t) = -\frac{P_1}{\Lambda}\left[1+p(t)\right]\int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau, \tag{3}$$

$$f(t) = 0, \qquad \text{for } t < 0. \tag{4}$$

The normalization and the increments are taken with respect to the equilibrium values at the operating power level. The minimum physical value of the variables is $p = c_i = -1$. The meaning of Eq. (3) is that reactivity feedback effects and external controls are expressed as a linear convolution integral of the incremental power

$$\int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau.$$

The kernel $f(t)$ must be such that no finite escape times exist, since no real reactor experiences finite escape times[1]. The convolution integral is equivalent to writing the feedback reactivity and the external controls as a linear combination, with constant coefficients, of the power and the other reactivity inducing variables. The constant coefficients are suitably defined coefficients of reactivity. The other variables are related to power by linear differential equations with constant coefficients. The remaining symbols in Eqs. (1)-(3) have their usual meaning.

Before proceeding with the stability analysis, it is convenient to define the following quantities:

$$q(j\omega,t) = \int_{-\infty}^{t} \exp\left[j\omega(t-\tau)\right] k(\tau)d\tau, \tag{5}$$

$$q^*(j\omega,t) = \int_{-\infty}^{t} \exp\left[-j\omega(t-\tau)\right] k(\tau)d\tau, \tag{6}$$

$$W(s) = \left(\Lambda s + \sum_i^m \frac{\beta_i s}{s+\lambda_i}\right)^{-1}$$

$$= \text{zero-power reactor transfer function}, \tag{7}$$

$$w(t) = \text{inverse transform of } W(s);\ w(t) = 0,\ t < 0, \tag{8}$$

$$F(s) = \text{Laplace transform of } f(t)$$

$$= \text{feedback transfer function}. \tag{9}$$

With these definitions, Eqs. (1)-(3) yield:

$$p(t) = \Lambda \int_{-\infty}^{t} w(t-\tau)k(\tau)d\tau$$

$$= \frac{\Lambda}{2\pi}\int_{-\infty}^{\infty} W(j\omega)q(j\omega,t)d\omega$$

$$= \frac{\Lambda}{2\pi}\int_{-\infty}^{\infty} W^*(j\omega)q^*(j\omega,t)d\omega, \tag{10}$$

$$\int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau = \frac{\Lambda}{2\pi}\int_{-\infty}^{\infty} F(j\omega)W(j\omega)q(j\omega,t)d\omega \tag{11}$$

$$k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(j\omega,t)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} q^*(j\omega,t)d\omega. \tag{12}$$

The derivation of Eqs. (10)-(12) is straightforward. It requires certain changes of order of integration which imply that:

$$\int_{-\infty}^{0} \left| p(\tau) \right| d\tau < \infty; \quad \int_{-\infty}^{0} \left| k(\tau) \right| d\tau < \infty. \tag{13}$$

III. CONDITION FOR STABILITY OF THE LINEAR APPROXIMATION

In order for the solutions of the linear approximation of Eqs. (1)-(3) to be stable, the roots of the characteristic equation

$$1 + P_1 W(s)F(s) = 0 \tag{14}$$

must lie in the left-half complex plane. It is assumed that this is true.

It is further assumed that, beyond a certain critical power level P_c, one or more of the roots of Eq. (14) move into the right-half complex plane and that the reactor model becomes linearly unstable. In other words, it is assumed that, for

$$P_c = a_c P_1;\ a_c > 1, \tag{15}$$

the equation

$$1 + a_c P_1 W(s)F(s) = 0 \tag{16}$$

admits roots on the $j\omega$ axis for the first time. The number a_c is a measure of the margin of linear stability with respect to the operating power level P_1.

IV. SUFFICIENT CONDITION FOR NONLINEAR ASYMPTOTIC STABILITY

Stability will be investigated here in the sense of Lyapunov. To this end, consider the scalar function:

$$V = p(t) - \ln[1+p(t)] - \frac{1}{2d^2}p^2(t)$$
$$+ \sum_i^m \frac{\beta_i}{\lambda_i\Lambda}\left\{c_i(t) - \ln[1+c_i(t)] - \frac{1}{2d^2}c_i^2(t)\right\}$$
$$+ \sum_i^m \frac{\beta_i}{\Lambda}\int_{-\infty}^t [p(\tau)-c_i(\tau)]^2\left\{\frac{1}{[1+p(\tau)][1+c_i(\tau)]} - \frac{1}{d^2}\right\}d\tau$$
$$+ \frac{b\Lambda}{a}\int_{-\infty}^t \frac{a-1-p(\tau)}{1+p(\tau)}k^2(\tau)d\tau\,, \tag{17}$$

where a, b, and d are positive numbers to be determined and a, $d > 1$. If $d \geq a$, the function V is positive definite in the region

$$d \geq a;\ -1 < p < a-1;\ -1 < c_i < a-1\,,\ i = 1,2,\ldots m. \tag{18a}$$

If $d < a$, the function V is positive definite in the region

$$d < a;\ -1 < p < d-1;\ -1 < c_i < d-1\,,\ i = 1,2,\ldots m. \tag{18b}$$

In addition, V admits continuous partial derivatives with respect to $p(t)$ and $c_i(t)$, and it is equal to zero only when $p = c_i = 0$, and when the two integrals are equal to zero, i.e., when $p_i = c_i = k = 0$. Note that, since the system of Eqs. (1)-(3) is autonomous, the function V is not explicitly dependent on time. It is convenient, however, to write the two integrals in Eq. (17) as functions of t to avoid the necessity for the definition of new variables and the consideration of partial derivatives with respect to these variables. Finally, it is assumed that all variables are well behaved in the interval $-\infty < t \leq 0$ so that the integrals are bounded in that interval.

The time derivative of V along the trajectories of the system of Eqs. (1)-(3) is given by

$$\frac{dV}{dt} = -\frac{P_1}{\Lambda}p(t)\int_{-\infty}^t f(t-\tau)p(\tau)d\tau - \frac{1}{d^2}p(t)k(t)$$
$$-\frac{b\Lambda}{a}k^2(t) - bP_1k(t)\int_{-\infty}^t f(t-\tau)k(\tau)d\tau\,. \tag{19}$$

If all the terms on the right-hand side of Eq. (19) are written as functions of $q(j\omega,t)$ by means of Eqs. (10)-(12), then it is found that (see Appendix)

$$\frac{dV}{dt} = -\frac{\Lambda}{4\pi^2}\left|\int_{-\infty}^{\infty}[\mathrm{Re}G(j\omega)]^{1/2}q(j\omega,t)d\omega\right|^2, \tag{20}$$

where

$$G(j\omega) = P_1\left|W(j\omega)\right|^2F(j\omega) + \frac{W^*(j\omega)}{d^2}$$
$$+\frac{b}{a}[1+aP_1W(j\omega)F(j\omega)]\,, \tag{21}$$

and provided that

$$d^2 = a;\ \ \mathrm{Re}G(j\omega) > 0. \tag{22}$$

Consequently, if there exist positive numbers a and $b(a>1)$ such that condition (22) is satisfied, the solutions of Eqs. (1)-(3) are asymptotically stable because V is a Lyapunov function with a negative definite time derivative. (Note that $dV/dt = 0$ only for $p = c_i = 0$. Indeed, suppose that this is not true. Then from Eq. (19) it is seen that $dV/dt = 0$ only for $k = 0$. But if $k = 0$ the only admissible solutions for p and c_i are $p = c_i = 0$.) In other words, given a reactor described by Eqs. (1)-(3), the operating power level P_1 is asymptotically stable for all initial perturbations that lie in the region (total quantities)

$$d^2 = a;\ \ P < dP_1;\ \ C_i < dC_{1i},\ \ i = 1,2,\ldots m\,, \tag{23}$$

provided that there exist positive numbers a and b that render the function $G(s)$ a positive real function without zeros on the $j\omega$ axis.

It is evident from the discussion in Sec. III that the number a must be:

$$1 < a < a_c. \tag{24}$$

Its exact value, as well as that of b, depends on the particular form of the feedback transfer function $F(s)$ and the zero-power reactor transfer function $W(s)$.

The criterion given by inequality (22) may also be interpreted as implying either that, for a given zero-power reactor transfer function $W(s)$ and a given desirable range of operating power levels, 0 to aP_1, a feedback transfer function $F(s)$ can be found that guarantees nonlinear asymptotic stability, or that, for given $W(s)$ and $F(s)$, a range of operating power levels, 0 to aP_1, can be found in which the reactor is nonlinearly asymptotically stable.

Several simplified criteria can be derived from inequality (22). For example, when $d^2 = a$, then:

$$G(j\omega) = \left[\frac{W^*(j\omega)}{a} + \frac{b}{a}\right]\left[1+aP_1W(j\omega)F(j\omega)\right]$$
$$= \frac{1}{a}\left|1+aP_1W(j\omega)F(j\omega)\right|^2$$
$$\times\frac{W^*(j\omega)+b}{1+aP_1W^*(j\omega)F^*(j\omega)}\,. \tag{25}$$

Therefore, requirement (22) is equivalent to

$$\mathrm{Re}\left[\frac{W(j\omega)+b}{1+aP_1W(j\omega)F(j\omega)}\right] > 0\,. \tag{26}$$

A special form of inequality (26) occurs when $b = 0$. Then the sufficient condition for asymptotic stability becomes:

$$\mathrm{Re}\left[\frac{W(j\omega)}{1+aP_1W(j\omega)F(j\omega)}\right] > 0\,. \tag{27}$$

This criterion implies that the reactor is asymptotically stable with respect to all initial perturbations in the region (23) if the reactor transfer function at power aP_1 is a positive real function without zeros on the $j\omega$ axis. In other words, the reactor is asymptotically stable if the transfer function at power aP_1 is like the input impedance of a RLC network without imaginary zeros.

Finally, another simplified criterion results for b very large, i.e., by considering only the last integral in the V function [Eq. (17)]. Then, the sufficient criterion for asymptotic stability reduces to:

$$\mathrm{Re}[1+aP_1W(j\omega)F(j\omega)] > 0. \tag{28}$$

V. COMPARISONS WITH EXISTING CRITERIA OF STABILITY

Welton's sufficient criterion for stability[5] of the solutions of the system of Eqs. (1)-(3) is:

$$\mathrm{Re}F(j\omega) > 0. \tag{29}$$

It guarangees asymptotic stability over an infinite range of operating power levels, and it does not include the delayed neutron precursor constants. Welton's criterion is a special case of inequality (22) for $a = d = \infty$, $b = 0$. Even for an infinite power range, however, the simplified requirement (28) may, in certain cases, be less restrictive than condition (29). Indeed, the meaning of Welton's criterion is that:

$$-90^\circ < \arg F(j\omega) < 90^\circ. \tag{30}$$

For $a \to \infty$, requirement (28) reduces to

$$\mathrm{Re}W(j\omega)F(j\omega) > 0\ . \tag{31}$$

Since $-90^\circ < \arg W(j\omega) < 0^\circ$, for $\omega > 0$, it is clear that for a leading $F(j\omega)$ inequality (30) is more restrictive than inequality (31).

Popov also derived a sufficient criterion for asymptotic stability over an infinite range of operating power levels[6]. The criterion is given by the relation

$$\mathrm{Re}F(j\omega)[(\alpha+\beta j\omega)/j\omega] > 0, \tag{32}$$

where α and β are arbitrary positive constants. If requirement (22) is rewritten in the form:

$$\mathrm{Re}F(j\omega)\left[|W(j\omega)|^2 + bW(j\omega)\right] > -\frac{1}{P_1}\mathrm{Re}\left[\frac{W^*(j\omega)}{d^2} + \frac{b}{a}\right], \tag{33}$$

it is seen that Inequality (33) for certain cases may be less restrictive than condition (32.) For $P_1 \to \infty$(or $a = d \to \infty$),the right-hand side of Inequality (33) is zero, but the multiplier of $F(j\omega)$ in the left-hand side is different than the factor $[(\alpha+\beta j\omega)/j\omega]$ in Inequality (32). For P_1 finite (or a finite), the right-hand side of Inequality (33) is nonpositive and hence less restrictive than that of Inequality (32).

In addition, Popov presented another criterion for asymptotic stability including a finite range of operating power levels and the delayed neutron precursor constants[7]. In terms of the present nomenclature the criterion reads:

$$\mathrm{Re}\left\{P_1\left|W(j\omega)\right|^2F(j\omega) + \frac{b}{a}\left[1 + aP_1W(j\omega)F(j\omega)\right]\right\} > 0\,. \tag{34}$$

It is derived by means of certain inequalities and the range of acceptable initial perturbations is given in an implicit form. This criterion is also a special case of and more restrictive than requirement (22). Indeed, for $d = \infty$, the two criteria are identical, for $d^2 = a$, however, requirement (22) is less restrictive because the term $\mathrm{Re}W^*(j\omega)/a$ is nonnegative.

Baran and Meyer derived another criterion for absolute stability[4] which is stated as

$$\mathrm{Re}W(j\omega)F(j\omega) > 0. \tag{35}$$

This criterion coincides with requirement (28) for $a \to \infty$, but it is more restrictive even than (28) for a finite range of operating power levels.

It is concluded that the sufficient condition for asymptotic stability stated by inequality (22) is the most general sufficient criterion that has been derived to date.

VI. EXPERIMENTAL INTERPRETATION OF FREQUENCY-DEPENDENT STABILITY CRITERIA

All the sufficient criteria for nonlinear stability discussed in Secs. IV and V involve only parameters characteristic of the linear approximation of Eqs. (1)-(3), i.e., only the zero-power reactor transfer function $W(s)$ and the feedback transfer function $F(s)$. For reactor systems representable by Eqs. (1)-(3), this is an important advantage. Linear characteristics can be measured by means of small-amplitude perturbation tests. Consequently, predictions about nonlinear stability can be made without overexciting the reactor system. In addition, procedures for identifying nonlinear systems are not yet fully developed. Therefore,

[5] T. A. WELTON, "Kinetics of Stationary Reactor Systems," *Proc. First U. N. Intern. Conf. Peaceful Uses At. Energy*, 5, 377 (1955).

[6] V. M. POPOV, "Absolute Stability of Nonlinear Systems of Automatic Control," *Automation Remote Control*, **22**, 857 (1962).

[7] V. M. POPOV, "A New Criterion for the Stability of Systems Containing Nuclear Reactors," *Rev. Electrotech. Energ.*, **A8**, 113 (1963).

criteria for nonlinear stability stated in terms of linear characteristics are essential.

To illustrate how experimental data from oscillation, auto- or cross-correlation tests can be used to predict nonlinear stability, consider the form of requirement (22) given by inequality (26). This form can be factored to read:

$$\mathrm{Re}\left[\frac{W(j\omega)}{1+aP_1W(j\omega)F(j\omega)}\right]\left[1+\frac{b}{W(j\omega)}\right] > 0\ . \qquad (36)$$

Note that the first bracket in inequality (36) is the reactor transfer function at power aP_1. Call this transfer function $H_a(j\omega)$. The second bracket in Inequality (36) depends only on the zero-power reactor transfer function $W(s)$ and the arbitrary constant b. For positive frequencies, the phase $\psi_b(\omega)$ of the second bracket varies between 0 and 90°. Hence, Inequality (36) is equivalent to

$$\phi_{2b}(\omega) = -90 - \psi_b(\omega) < \arg H_a(j\omega) < 90° - \psi_b(\omega) = \phi_{1b}(\omega), \qquad \text{for} \qquad \omega > 0. \qquad (37)$$

In other words, asymptotic stability is guaranteed if the phase of the reactor transfer function at power is between $\phi_{2b}(\omega)$ and $\phi_{1b}(\omega)$. This requirement can be readily implemented experimentally by means of the following procedure:

a) Measure the zero-power transfer function $W(j\omega)$. Use the data to compute the phases $\psi_b(\omega)$, $\phi_{1b}(\omega)$, and $\phi_{2b}(\omega)$ for several values $0 < b < \infty$. For a typical boiling water reactor, say EBWR[8], representative results for $\phi_{1b}(\omega)$ and $\phi_{2b}(\omega)$ are shown in Fig. 1.

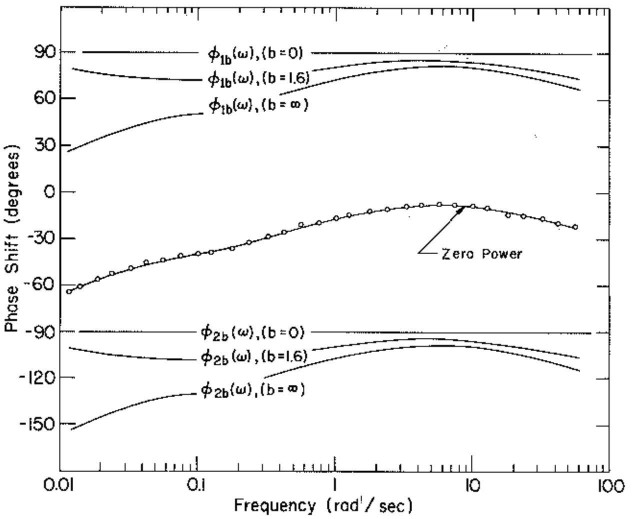

Fig. 1. Measured phase shift of EBWR zero-power transfer function $W(j\omega)$, and phase shift of $1+[b/W(j\omega)]$ for different values of b.

[8]W. C. LIPINSKI, A. HIRSH, C. HSU, and G. POPPER, "EBWR Reactor Transfer Function Measurements," ANL 6703, 268, Argonne National Laboratory, Argonne, Ill. (1964).

b) Measure the reactor transfer function $H_1(j\omega)$ at power P_1 and plot the ratio $H_1(j\omega)/W(j\omega) = 1/[1+P_1W(j\omega)F(j\omega)]$ on a Nichols chart. Such a plot for EBWR at 40 MW is shown in Fig. 2. From the Nichols chart compute the phase of different $H_a(j\omega)/W(j\omega)$ by simply moving the zero decibel level by steps corresponding to different a's. Add to the results the phase of $W(j\omega)$ to find the phase of several $H_a(j\omega)$'s. Plot the phase of each $H_a(j\omega)$ on Fig. 1 and compare it with pairs of curves $\phi_{1b}(\omega)$ and $\phi_{2b}(\omega)$, to satisfy inequalities (37). Choose that pair of curves, i.e., that value of b, which permits the maximum value a_m of a without violating requirements (37). This maximum value a_m gives the range, 0 to a_mP_1, of operating power levels over which the reactor is nonlinearly, asymptotically stable. Similarly, $(a_m-1)P_1$ gives the range of power perturbations for which the equilibrium power level P_1 is nonlinearly asymptotically stable. The ratio a_m/a_c is a measure of the range over which the reactor is nonlinearly asymptotically stable with regard to that for linear stability.

Step (b) has not been completed for EBWR because its feedback transfer function varies with the operating power level. In such cases one might proceed as in (c) below.

c) If the kernel $f(t)$, i.e., the feedback transfer function $F(s)$ is not the same at all operating power levels, then the dynamics of the reactor are not strictly representable by Eqs. (1)-(3). More specifically, $k(t)$ is not given by Eq. (3) but by a relation of the form:

$$k(t) = -\frac{P_1}{\Lambda}\left[1+p(t)\right]\left\{\int_{-\infty}^{t} f_1(t-\tau)p(\tau)d\tau + F_2[p(\tau)]\right\}, \qquad (39)$$

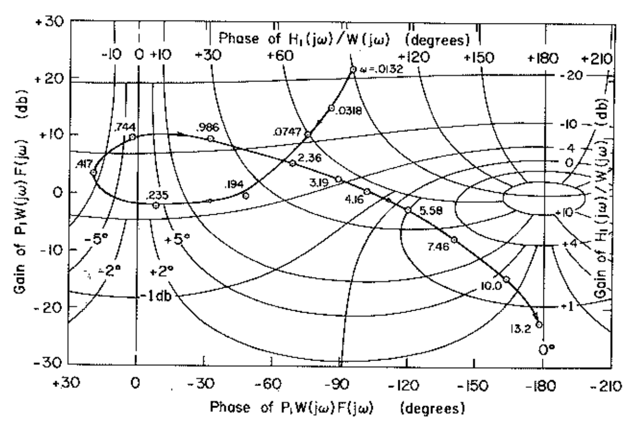

Fig. 2. Measured EBWR reactor transfer function at 40 MW over EBWR zero-power transfer function plotted on a Nichols chart.

where $f_1(t)$ is determined by the particular operating power level P_1 and $F_2[p(\tau)]$ is a nonlinear functional. Some of the properties of $F_2[p(\tau)]$ have been discussed in previous work[1]. For the purposes of this paper, assume that the characteristic properties of the reactor system are weak functions of the operating power level and that for the power range of interest:

$$\delta = \left| F_2[p(\tau)] \Big/ \int_{-\infty}^{t} f(t-\tau)p(\tau)d\tau \right| \ll 1. \tag{40}$$

Thus, the time derivative of the scalar function V [Eq. (17), $d^2 = a$] along the trajectories of Eqs. (1), (2), (39) is:

$$\begin{aligned}\frac{dV}{dt} &= -\frac{P_1}{\Lambda}\, p(t) \int_{-\infty}^{t} f_1(t-\tau)p(\tau)d\tau - \frac{1}{a}\, p(t)k(t)\\ &\quad - \frac{b\Lambda}{a}\, k^2(t) - bP_1 k(t) \int_{-\infty}^{t} f_1(t-\tau)p(\tau)d\tau\\ &\quad - \frac{P_1}{\Lambda}\left[p(t) + b\Lambda k(t)\right] F_2[p(\tau)]\\ &= -\frac{a_1P_1}{\Lambda}\, p(t) \int_{-\infty}^{t} f_1(t-\tau)p(\tau)d\tau - \frac{1}{a}\, p(t)k(t)\\ &\quad - \frac{b\Lambda}{a}\, k^2(t) - ba_1P_1 k(t) \int_{-\infty}^{t} f_1(t-\tau)p(\tau)d\tau,\end{aligned} \tag{41}$$

where $a_1 = 1 \pm \delta$ and it is a number close to unity and δ is given by Eq. (40). This derivative can be reduced to a form identical to that of Eq. (20) because Eqs. (10)-(12) are not affected by Eq. (39). The function $G(j\omega)$, however, is presently given by the relation:

$$G(j\omega) = a_1P_1 \left|W(j\omega)\right|^2 F_1(j\omega) + \frac{W^*(j\omega)}{a} + \frac{b}{a}\left[1 + aa_1P_1W(j\omega)F_1(j\omega)\right], \tag{42}$$

where $F_1(s)$ is the Laplace transform of $f_1(t)$. Thus, the sufficient condition for nonlinear asymptotic stability is equivalent to requirement (22), where P_1 is replaced formally by a_1P_1.

This discussion suggests that, in reactors with smoothly varying characteristics, an estimate of the maximum power which is nonlinearly, asymptotically stable can be established by means of small amplitude tests by using inequalities (37) as follows: Assume that a_1 is a constant essentially equal to unity. Measure the reactor transfer function at several operating power levels. Superimpose the phases of these transfer functions on the $\phi_{ib}(\omega)$ diagrams. Typical phase curves for EBWR at different power levels are shown in Fig. 3. Estimate the maximum of the operating power level which is asymptotically stable by extrapola-

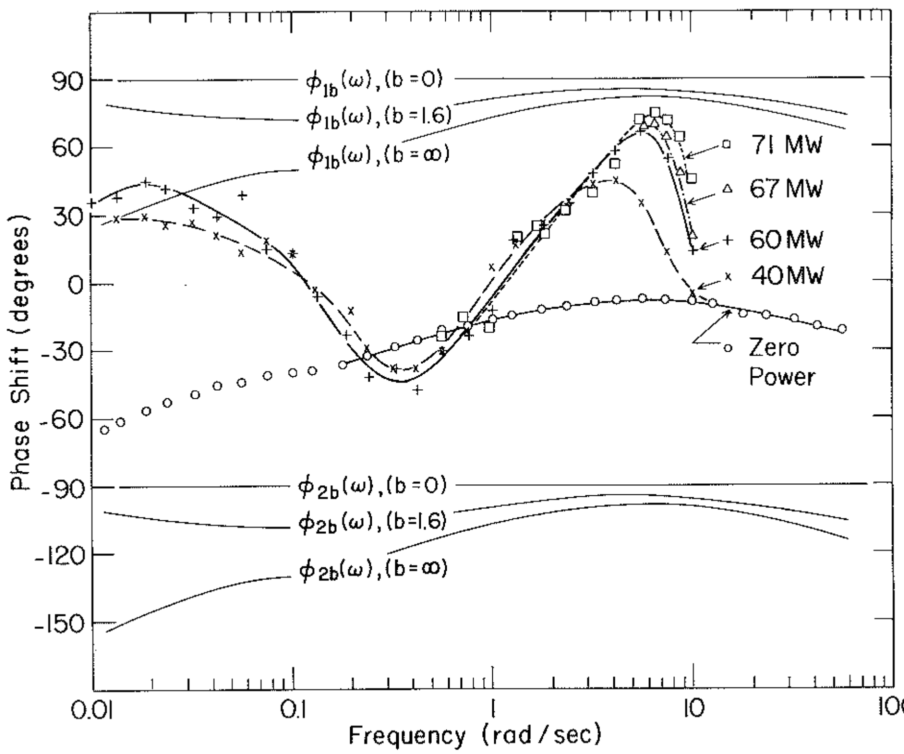

Fig. 3. Measured phase shift of EBWR at different power levels superimposed on $\phi_{1b}(\omega)$ and $\phi_{2b}(\omega)$ plots.

tion. More specifically, note that in the example of Fig. 3 the shape of the phase curves indicates that the best choice of b is zero and that the crucial points with regard to inequalities (37) are at the resonance around 6 radian/sec. Thus, plot the operating power level vs phase at this resonance and extrapolate the resulting curve to 90° (Fig. 4). For EBWR the extrapolation yields a maximum power P_{max} of 98 or 108 MW, depending on how the experimental data are employed. In view of the uncertainties which always accompany experimental results, it is not very important to find the exact value of a_1, provided that Eq. (40) is satisfied.

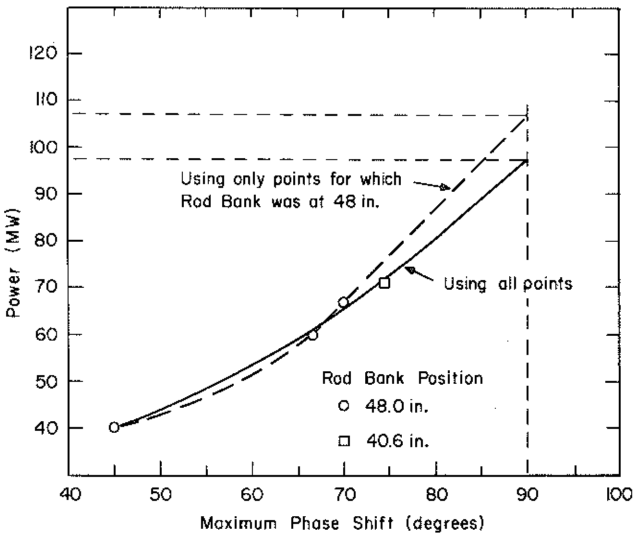

Fig. 4. Measured EBWR operating power level versus reactor transfer function phase shift at the resonance around 6 rad/sec.

The above procedure is remindful of criticality measurements where the critical mass is established by means of subcritical measurements. Similarly, an estimate of the maximum power level, at which the reactor is nonlinearly, asymptotically stable, is found by performing small-amplitude perturbation (linear) measurements at power levels smaller than P_{max}.

VII. CONCLUSIONS

New sufficient criteria for nonlinear asymptotic stability of reactor systems with linear reactivity feedback have been derived. The criteria are shown to be more general and less restrictive than all other similar criteria that have been derived to date. The criteria are also extended to reactor systems with nonlinear feedback.

A systematic procedure is outlined for the experimental interpretation of the criteria by means of small amplitude perturbation tests, i.e., by means of nonhazardous tests. The procedure is meaningful only for reactor systems with 'smooth' characteristics, i.e., reactor systems without threshold effects which may be excited only by large perturbations. The reason for the restriction is that threshold effects are not included in Eqs. (1)-(3) or (1), (2), (39), and (40).

APPENDIX

The time derivative dV/dt [Eq. (19)] is a real function. By repeated and suitable use of Eqs. (10)-(12), this derivative can be written as:

$$\frac{dV}{dt} = -\frac{\Lambda}{8\pi^2}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 K(\omega_1,\omega_2) \times q^*(j\omega_1,t)q(j\omega_2,t), \qquad (A\text{-}1)$$

where

$$K(\omega_1,\omega_2) = G(j\omega_1,j\omega_2) + G^*(j\omega_2,j\omega_1), \qquad (A\text{-}2)$$

$$G(j\omega_1,j\omega_2) = P_1W^*(j\omega_1)W(j\omega_2)F(j\omega_2) + \frac{W^*(j\omega_1)}{d^2} + \frac{b}{a}\left[1+aP_1W(j\omega_2)F(j\omega_2)\right]. \qquad (A\text{-}3)$$

When $d^2 = a$, Eq. (A-3) can be written as:

$$G(j\omega_1,j\omega_2) = A_1^*B_2 ;$$
$$A_i = b + W(j\omega_i) ;$$
$$B_i = \frac{1}{a} + PW(j\omega_i)F(j\omega_i) . \qquad (A\text{-}4)$$

Hence, the kernel $K(\omega_1,\omega_2)$ can be written as:

$$K(\omega_1,\omega_2) = A_1^*B_2 + A_2B_1^* , \qquad (A\text{-}5)$$

and it satisfies the relations:

$$K(\omega,\omega) > 0; \qquad K(\omega_1,\omega_2) = K^*(\omega_2,\omega_1) , \qquad (A\text{-}6)$$

provided that $\mathrm{Re}G(j\omega) > 0$, i.e., condition (22) is satisfied. Furthermore, it is readily shown that:

$$|K(\omega_1,\omega_2)|^2 = (A_1^*B_1+A_1B_1^*)(A_2^*B_2+A_2B_2^*) + |A_1B_2-A_2B_1|^2$$
$$= 4\mathrm{Re}G(j\omega_1)\,\mathrm{Re}G(j\omega_2) + C^2(\omega_1,\omega_2); \qquad (A\text{-}7)$$

$$C(\omega_i,\omega_j) = (-1)^j|A_iB_j-A_jB_i| ;$$
$$C(\omega,\omega) = 0 ;$$
$$C(\omega_1,\omega_2) = -C(\omega_2,\omega_1) . \qquad (A\text{-}8)$$

In view of conditions (A-6), the kernel $K(\omega_1,\omega_2)$ must be given by the relation:

$$K(\omega_1,\omega_2) = 2[\mathrm{Re}G(j\omega_1)\,\mathrm{Re}G(j\omega_2)]^{1/2} + jC(\omega_1,\omega_2). \qquad (A\text{-}9)$$

If Eq. (A-4) is replaced in Eq. (A-1), it is found that:

$$\frac{dV}{dt} = -\frac{\Lambda}{4\pi^2}\left|\int_{-\infty}^{\infty}[\mathrm{Re}G(j\omega)]^{1/2}\,q(j\omega,t)d\omega\right|^2, \qquad \text{(A-10)}$$

because the double integral corresponding to $C(\omega_1,\omega_2)$ is equal to zero since $C(\omega_1,\omega_2)=-C(\omega_2,\omega_1)$. This result is identical with Eq. (20), and it is derived under the provision that condition (22) is satisfied.

It can be shown by means of the theory of reproducing kernels[9] that $dV/dt < 0$ also for $\mathrm{Re}G(j\omega) > 0$, $d^2 \gtrless a$. The maximum allowable power level for asymptotic stability, however, is given by condition (22), i.e., $\mathrm{Re}G(j\omega) > 0$, $d^2 = a$. This can be seen by considering aP_1 such that condition (22) yields $d^2 = a = 1 + \epsilon$, where ϵ is very small, and then investigating the consequences of requiring $\mathrm{Re}G(j\omega) > 0$ for $d^2 \gtrless a$.

[9] N. ARONSZAJN, "The Theory of Reproducing Kernels," *Trans. Am. Math. Soc.*, 68, 337 (1950).

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