Application of the Natural Mode Approximation to Space-Time Reactor Problems

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A space-dependent reactor kinetics approximation, called the Natural Mode Approximation (NMA), has been applied to the calculation and interpretation of reactor dynamic experiments. The NMA is based on a modal expansion technique where the space- and time-dependent reactor variables are approximated by a series of products of time-dependent coefficients and space-dependent expansion modes. The modes are the eigenvectors of a linear operator derived from the complete set of equations describing the reactor system at an initial reference condition. A pair of computer codes, MUDMO-II and SYNSIG, are used to synthesize approximate modes in two-dimensional systems without feedback.

An oscillation test is proposed which may be used to verify key parameters of the NMA. The experimental technique is described and applied to both numerical and actual measurements. In addition, it is shown how a natural mode expansion may be used to interpret standard dynamic experiments when the observations are functions of space and time.

The results of calculations of kinetic problems are compared with those of independent calculations which are considered to be exact. Good agreement is established. It is shown that the flux tilting following a localized perturbation is a sensitive function of the relative magnitudes of the measurable parameters of the NMA. The novel idea of "correction modes" is introduced which increases the accuracy of a low-order NMA without appreciable increase in computation time.

I. INTRODUCTION

The purpose of this paper is to indicate how the Natural Mode Approximation (NMA) may be used in thermal reactors that experience non-negligible flux shape changes: 1) to interpret oscillation tests; and 2) to calculate a number of practical transients.

Different approximations for dynamic spacedependent reactor problems are reviewed by Kaplan et al.,¹ which also includes a very long list of publications. For a given accuracy, we feel that an approximation is "best" when: 1) it involves experimentally verifiable parameters; 2) it is suitable for a broad class of problems; and 3) it requires a reasonable computational effort. For the problems of interest to this paper, we show that the NMA has the above features.

The NMA is well known from the work of several others. $^{2-4}$ It is derived from an expansion of the neutron densities and other dependent

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¹S. KAPLAN, A. F. HENRY, S. G. MARGOLIS, and J. J. TAYLOR, "Space-Time Reactor Dynamics," *Proc. Intern. Conf. Peaceful Uses At. Energy*, P/271, Vol. 4, 41 (1964).

²R. COHEN, "Some Topics in Reactor Kinetics," Proc. Intern. Conf. Peaceful Uses At. Energy, P/629, Vol. 11, 302 (1958).

³S. KAPLAN, "The Property of Finality and the Analysis of Problems in Reactor Space-Time Kinetics by Various Modal Expansions," *Nucl. Sci. Eng.*, 9, 357 (1961).

⁴A. F. HENRY, "The Application of Inhour Modes to the Description of Nonseparable Reactor Transients," *Nucl. Sci. Eng.*, **20**, 338 (1964).

variables in a finite series of products of unknown time-dependent coefficients, and known spacedependent expansion vectors. The expansion vectors are the eigenvectors of a linear operator formed from the complete set of equations describing the reactor system at an initial, reference condition. The eigenvectors are called the natural modes of the reference system. The associated eigenvalues, which appear prominently in the equations describing the unknown timedependent coefficients, are called the parameters of the NMA.

The contributions of this paper are the proposal and development of the following ideas: 1) key parameters (particular natural mode eigenvalues) of the NMA may be verified experimentally by means of small signal oscillation tests; 2) the verifiable parameters are sensitive indicators of the flux tilting which follows a localized perturbation; 3) natural modes can be used to interpret meaningfully the readings of a number of neutron detectors when each reading depends upon the position of the detector; and 4) the NMA can be used to calculate efficiently a number of important reactor transients. In addition, it is shown that two-dimensional natural modes may be constructed by means of a "synthesis" technique.⁵

The paper is organized as follows: First, the formalism of the NMA is reviewed briefly. Second, the NMA is used to interpret oscillation tests. Some important parameters of the NMA are experimentally verified. Measurements of subcriticality and stability are interpreted meaningfully in the presence of spatial effects. Third, a correlation is given which relates the measurable parameters to flux tilting. Fourth, the NMA is applied to the calculation of fast transients with and without feedback. Finally, preliminary results of calculations of two-dimensional natural modes are presented.

II. REVIEW OF THE FORMALISM OF THE NMA

For simplicity, the review of the formalism of the NMA is presented for a reactor without feedback. For more complete discussions, see Refs. 3, 6, 7, and 8.

⁵S. KAPLAN, "Some New Methods of Flux Synthesis," *Nucl. Sci. Eng.*, 9, 22 (1962).

⁶L. R. FOULKE, "A Modal Expansion Technique for Space-Time Reactor Kinetics," PhD Thesis, Massachusetts Institute of Technology, Dept. of Nuclear Engineering, Cambridge, Massachusetts (August 1966).

⁷D. WIBERG, "Optimal Feedback Control of Spatial Xenon Oscillations in a Nuclear Reactor," PhD Thesis, California Institute of Technology, Pasadena, California (1964).

⁸S. KAPLAN and J. B. YASINSKY, "Natural Modes of the Xenon Problem with Flow Feedback - An Example," *Nucl. Sci. Eng.*, **25**, 430 (1966). In the framework of the G-group diffusion – theory approximation, the space- and time-dependent behavior of a reactor, with I groups of delayed-neutron precursors and without feedback, is described by the matrix relation

$$\frac{\partial \boldsymbol{\psi}(\boldsymbol{x},t)}{\partial t} = \left[H(\boldsymbol{x},t) \right] \boldsymbol{\psi}(\boldsymbol{x},t) + \boldsymbol{S}(\boldsymbol{x},t) \quad , \tag{1}$$

where

$$\psi(\mathbf{x},t) \equiv \operatorname{col} [\mathbf{N},\mathbf{C}] ,$$
$$\mathbf{N} \equiv \operatorname{col} [N^{(1)}(\mathbf{x},t), N^{(2)}(\mathbf{x},t), \dots, N^{(G)}(\mathbf{x},t)] ,$$

and

$$C = \operatorname{col} \left[C^{(1)}(x,t), C^{(2)}(x,t), \ldots, C^{(l)}(x,t) \right] .$$

The g'th neutron-group density is denoted by $N^{(g)}(\boldsymbol{x},t)$, and the *i*'th precursor density is denoted by $C^{(i)}(\boldsymbol{x},t)$. The $K \times K$ matrix operator $[H(\boldsymbol{x},t)]$ consists of all the production and destruction operators, and K = G + I. The column vector $S(\boldsymbol{x},t)$ contains all external sources.

One method of finding an approximate solution of Eq. (1) is to proceed as follows: Consider a steady-state reference condition for which Eq. (1) becomes

$$[H_0(x)]\psi_0(x) + S_0(x) = 0 .$$
 (2)

Define the eigenvectors $\boldsymbol{\psi}_{mk}(\boldsymbol{x})$ of the eigenvalue problem

$$[H_0(\boldsymbol{x})] \boldsymbol{\psi}_{mk}(\boldsymbol{x}) = \omega_{mk} \boldsymbol{\psi}_{mk}(\boldsymbol{x})$$
(3)

as the natural modes of Eq. (2). Expand the solution vector $\psi(x,t)$ into a finite series of the form

$$\psi(x,t) = \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(t) \psi_{mk}(x) \quad . \tag{4}$$

Substitute this series into Eq. (1), multiply both sides of the equation by $M \cdot K$ solutions of the adjoint relation

$$[H_0^*(\boldsymbol{x})] \boldsymbol{\psi}_{nj}^*(\boldsymbol{x}) = \omega_{nj} \boldsymbol{\psi}_{nj}^*(\boldsymbol{x}) \quad , \qquad (5)$$

and integrate^a each result over all x. The result of this procedure is a set of $M \cdot K$, coupled, ordinary differential equations which may be written as

$$\frac{d\mathbf{A}}{dt} = \operatorname{diag}\left[\omega\right]\mathbf{A} + \left[P\right]\mathbf{A} + \mathbf{S} , \qquad (6)$$

^aThe Dirac bracket notation $\langle A, B \rangle$ is used to denote integration of the scalar product of the row vector A^{T} , and the column vector **B** over the volume of the reactor. The notation $\langle AB \rangle$ is used to denote integration of the product of two scalars A and B over the volume of the reactor.

where

$$A = \operatorname{col} [A_{11}(t), A_{12}(t), \dots, A_{MK}(t)] ,$$

$$S = \operatorname{col} [S_{11}(t), S_{12}(t), \dots, S_{MK}(t)] ,$$

$$S_{mk}(t) = \langle \psi_{mk}^{*}(x), S(x,t) \rangle / \langle \psi_{mk}^{*}(x), \psi_{mk}(x) \rangle ,$$

and diag $[\omega]$ is an $M \cdot K \times M \cdot K$ diagonal matrix with elements ω_{mk} . The perturbation matrix [P] has elements

$$P_{\mu\gamma} = \frac{\langle \psi_{nj}^*(\boldsymbol{x}), [H(\boldsymbol{x},t) - H_0(\boldsymbol{x})] \psi_{mk}(\boldsymbol{x}) \rangle}{\langle \psi_{nj}^*(\boldsymbol{x}), \psi_{nj}(\boldsymbol{x}) \rangle}$$

where

 $\mu = (n - 1)K + j$ for n = 1, 2, ..., M and j = 1, 2, ..., Kand

$$\gamma = (m-1)K + k$$
 for $m = 1, 2, ..., M$ and $k = 1, 2, ..., K$.

In the derivation of Eq. (6) it has been assumed that

$$\langle \psi_{nj}^*(\mathbf{x}), \psi_{mk}(\mathbf{x}) \rangle = 0 \text{ for } \omega_{mk} \neq \omega_{nj} ,$$

 $\langle \psi_{mk}^*(\mathbf{x}), \psi_{mk}(\mathbf{x}) \rangle \neq 0 ,$

and that $\psi_{mk}(x)$ and $\psi_{nj}^*(x)$ satisfy the same homogeneous boundary conditions as $\psi_0(x)$ and $\psi_0^*(x)$, respectively.

The result of the above procedure is a set of equations [Eq. (6)] which is referred to as the M'th order NMA. These equations are solved for the expansion coefficients $A_{mk}(t)$. The $A_{mk}(t)$'s are then used in Eq. (4) to construct an approximate solution. In essence, a problem [Eq. (1)] with K-dependent and four independent variables is reduced to another with $M \cdot K$ dependent and one independent variables. The motivation is that the latter problem may be easier to solve numerically.

There exist computer codes for the calculation of natural modes of one-dimensional reactor systems both with^{7,9} and without^{6,10-12} xenon feedback. The problem of calculation of natural modes of two-dimensional systems is discussed in Sec. IV of this paper.

To facilitate the developments which follow, it

is desirable to summarize the properties of the one-dimensional solutions of Eq. (3), and, thus, introduce a terminology which may be used to discuss the natural modes. It is well known^{1,4,12} that, for a reactor with G neutron and I precursor groups, the natural modes come in clusters of K(=G+I). The K modes of a cluster have components of similar, but not necessarily identical, shape. In general, the modes become more oscillatory in space as the cluster index m increases. For this reason, the modes of the m'th cluster are referred to as the modes of the m'th spatial harmonic. Figure 1 represents a typical eigenvalue spectrum of the first five spatial harmonics of a nonuniform, one-dimensional, thermal-reactor model with two neutron (G=2)and one precursor (I = 1) groups. The following features of this figure are generally true. The Ialgebraically largest eigenvalues associated with the m'th spatial harmonic are of the order of the precursor decay constants. They are called the delayed-neutron eigenvalues and the corresponding modes are called the delayed-neutron modes. The remaining G eigenvalues associated with the m'th spatial harmonic have larger magnitudes. They are called the prompt-neutron eigenvalues and the corresponding modes are called the prompt-neutron modes. As the index m increases, the delayed-neutron eigenvalues approach the values^b $-\lambda_i$, i = 1, 2, ..., I, while the promptneutron eigenvalues approach $-\infty$.

Of the *G* prompt-neutron eigenvalues of the *m*'th spatial harmonic, there is one of algebraically largest magnitude. This eigenvalue may be associated with a relaxation time of the asymptotic energy distribution of neutrons in the *m*'th harmonic. It is called the prompt thermal-neutron eigenvalue of the *m*'th spatial harmonic and it is denoted by the subscript k = p. Each ω_{mp} is a sensitive function of the thermal-neutron group constants and the geometrical arrangement of the core materials. It is shown subsequently that for small values of *m*, the ω_{mp} 's may be: 1) measured by means of a nonhazardous oscillation test; and 2) used to indicate the susceptibility of a reactor to flux tilting.

The other (G - 1) prompt-neutron eigenvalues associated with the *m*'th spatial harmonic are called prompt epithermal-neutron eigenvalues. They effectively represent the decay constants of energy transients in the multigroup approximation. They are of no particular interest to this paper.

⁹R. L. EWEN, "Calculation of Complex Natural Modes for Spatial Xenon Oscillations and Comparison with a Simple Approximation," *Trans. Am. Nucl. Soc.*, 5, 179 (1962).

¹⁰R. L. EWEN, "MULE - A FORTRAN Program for the Calculation of Three Types of Overtone Modes," WAPD-TM-369, Westinghouse Electric Corp. (1963).

¹¹C. A. PRESKITT and E. A. NEPHEW, "Transients Induced by Pulsed Neutron Sources in Multiregion Reactors," ORNL-TM-993, Oak Ridge National Laboratory (1965).

¹²T. GOZANI, "The Concept of Reactivity and its Application to Kinetic Measurements," *Nukleonik*, 5, 55 (1963).

^bFor a reactor model with J feedback variables it is possible to distinguish I delayed-neutron eigenvalues plus J feedback eigenvalues for each m. The behavior of the eigenvalues as a function of m in this case is beyond the scope of this paper.



Fig. 1. Eigenvalue spectrum of a nonuniform, slab reactor in which two neutron and one precursor groups are considered.

III. APPLICATIONS

III.1. Interpretation of Oscillation Tests

Measurement of the Prompt Thermal-Neutron Eigenvalues ω_{mp} .

Let a reactor, operating initially at steady state, be perturbed by a small, localized, thermalneutron absorber which oscillates sinusoidally at a frequency ω . After some time, during which the transients die out, the neutron density also oscillates sinusoidally at the same frequency as, but different amplitude and phase than, the absorber. The oscillating portion of the reading of a thermal-neutron detector, placed at location x_i , is of the form $Re[r_i(j\omega) \exp(j\omega t)]$. If this behavior is assumed to be describable by M spatial harmonics of the NMA, then it is readily shown (See Appendix A) that

$$\gamma_i(j\omega) \simeq \sum_{m=1}^M \sum_{k=1}^K d_{mk}^{(i)} A_{mk}(j\omega) \quad , \tag{7}$$

$$d_{mk}^{(i)} = \langle v_2 \sigma_i^{(2)} (\mathbf{x}) N_{mk}^{(2)} (\mathbf{x}) \rangle , \qquad (8)$$

and

$$A_{mk}(j\omega) = c_{mk}(j\omega - \omega_{mk})^{-1} , \qquad (9)$$

where $\sigma_i^{(2)}(\mathbf{x})$ is the response function of the *i*th thermal-neutron detector, $N_{mk}^{(2)}(\mathbf{x})$ is the thermal-neutron component of the *k*th natural mode of the *m*th spatial harmonic, v_2 is the thermal-neutron

speed, and c_{mk} is a constant whose value depends on the size and the location of the oscillator.

Equation (7) can be rewritten in the form

$$r_i(j\omega) \simeq \sum_{m=1}^N d_{mp}^{(i)} A_{mp}(j\omega) + e_{Ni}(j\omega) + e_i(j\omega) \quad , \quad (10)$$

where

$$e_{Ni}(j\omega) \equiv \sum_{m=N+1}^{M} d_{mp}^{(i)} A_{mp}(j\omega) \quad , \tag{11}$$

and

$$e_i(j\omega) = \sum_{m=1}^M \sum_{\substack{k=1\\k\neq p}}^K d_{mk}^{(i)} A_{mk}(j\omega) \quad .$$
(12)

In the form of Eq. (10), $r_i(j\omega)$ may be interpreted as consisting of three parts:

1) The contributions from the prompt thermalneutron modes with the N algebraically largest eigenvalues, ω_{mp} , $m = 1, 2, \ldots, N$.

2) The contribution $e_{Ni}(j\omega)$ from the remaining M-N, non-negligible prompt thermal-neutron modes. If the detector is small so that

$$\sigma_i^{(2)}(\mathbf{x}) = \sigma_i^{(2)} \,\delta(\mathbf{x} - \mathbf{x}_i) \quad , \tag{13}$$

then $e_{Ni}(j\omega)$ is an oscillatory function of the detector location x_i , and has at least N zero crossings.

3) The contribution $e_i(j\omega)$ from all the delayed and all the prompt epithermal-neutron modes. For oscillation frequencies that are larger and smaller than the magnitude of the delayed and prompt epithermal-neutron eigenvalues, respectively, this contribution is negligible in thermal reactors.

In general, for the frequency range specified above, if N small detectors are placed at the locations at which the sum $e_{Ni}(j\omega) + e_i(j\omega)$ is very small, the oscillatory readings of these detectors are given to a good approximation by the relations

$$r_i(j\omega) \simeq \sum_{m=1}^N d_{mp}^{(i)} A_{mp}(j\omega) \quad \text{for } i = 1, 2, \ldots, N \quad . \quad (14)$$

These relations can also be written in matrix form as

$$\boldsymbol{r} \simeq [\boldsymbol{v}] \boldsymbol{A}'$$
, (15)

where $\mathbf{r} = \operatorname{col}[r_1(j\omega), \ldots, r_N(j\omega)], A' = \operatorname{col}[A_{1p}(j\omega), \ldots, A_{Np}(j\omega)], and [v] is an <math>N \times N$ matrix with elements $u_{im} = d_{mp}^{(i)}$. If $N_{mp}^{(2)}$ for $m = 1, 2, \ldots, N$, and σ_i for $i = 1, 2, \ldots, N$ are known, Eq. (15) can be solved for A' in the range of oscillation frequency under consideration. Thus, the prompt thermal-neutron eigenvalues ω_{mp} for $m = 1, 2, \ldots, N$, may be estimated from the break frequencies of the $A_{mp}(j\omega)$'s [see Eq. (9)].

The maximum number of ω_{mp} 's, which can be verified experimentally in this fashion, is limited by the maximum frequency ω_{max} of the oscillator. The reasons are that ω_{mp} approaches $-\infty$ as mincreases, and that the verification of a particular ω_{mp} presumes that it is within the range of oscillation frequency of the absorber. Consequently, if

$$|\omega_{Dp}| < \omega_{\max} < |\omega_{D+1,p}| \quad , \tag{16}$$

then any number N of detectors smaller than or equal to D can be used to derive good estimates of an equal number of the algebraically largest, prompt thermal-neutron eigenvalues.

The best locations for N detectors [i.e., those at which the sum $e_{Ni}(j\omega) + e_i(j\omega)$ equals zero for i = 1, 2, ..., N] cannot be specified exactly in a complex reactor core. There are, however, approximate techniques which can be used to derive reasonable results. For example, as a first guess, the N detectors may be located at the zeros of the lowest order, non-negligible spatial harmonic in $e_{Ni}(j\omega)$. Specifically, if it is found that

$$c_{N+1,p} \neq 0$$
,

then a first estimate of $N \omega_{mp}$'s may be obtained by locating N detectors at x_i 's for which

$$N_{N+1,p}^{(2)}(\boldsymbol{x}_i) = 0 \quad . \tag{17}$$

These first estimates are often quite good but can be further improved. Results from many examples, which could be analyzed both exactly and by means of the NMA, indicate that the estimates improve if the detectors are displaced a small distance away from the zeros of $N_{N+1,p}^{(2)}(\mathbf{x})$ in the direction towards the oscillator.^c On the basis of this experience, the following rule of thumb has been found to be helpful: To verify the ω_{mp} 's associated with N prompt thermal-neutron modes, displace N detectors a small distance away from N zeros of the lowest order, non-negligible mode in $e_{Ni}(j\omega)$ in the direction towards the oscillator. These locations are satisfactory when the phase of the inferred $A_{1p}(j\omega)$ approaches -90° for frequencies $\omega \gg |\omega_{1p}|$.

Results

The procedure outlined above has been applied both to "numerical experiments" and to actual measurements.

For one of the numerical oscillation tests, a uniform slab reactor of extrapolated width a =200 cm is considered. A plane thermal-neutron absorber is oscillated at position $x_0 = a/4$ over the frequency range of 10 to 1000 rad/sec. Table I shows the nuclear constants of this reactor along with the calculated, prompt thermal-neutron eigenvalues of the first four spatial harmonics. It follows that only three (N = 3) of these eigenvalues can be verified experimentally with the available frequency range of the oscillator.

For this simple example, the natural modes are sinusoidal functions of position. In addition, the space- and frequency-dependent neutron density may be evaluated from a closed form solution;

TABLE I Nuclear Constants and Eigenvalues for the Numerical Oscillation Test

Nuclear Constants						
width = $\Sigma_{a2} =$ $\nu \Sigma_{f2} =$ $D_2 =$ $v_2 =$ $\beta =$	200 cm 0.02775 cm^{-1} 0.0282 cm^{-1} 0.804 cm $2.48 \times 10^5 \text{ cm/sec}$ 0.0074	$\Sigma_r = 0.03776 \text{ cm}^{-1}$ $\Sigma_{a1} = 0.0 \text{ cm}^{-1}$ $\nu \Sigma_{f1} = 0.0 \text{ cm}^{-1}$ $D_1 = 1.313 \text{ cm}$ $v_1 = 4.06 \times 10^6 \text{ cm/sec}$ $\lambda = 0.0138 \text{ sec}$				
	Inferred Eigenvalues, sec ⁻					
Spatial Harmonic Index, m	Actual Eigenvalue, sec ⁻¹	Detectors at Zeros of Fifth Harmonic	Detectors Displaced Towards Oscillator			
1 2 3 4	-49.1 -355 -850 -1514.	-48.9 -348 -788 	-49.2 -358 -860 			

^c The zero crossings of $e_{Ni}(j\omega) + e_i(j\omega)$ are displaced a small distance from those of $N_{N+1,p}^{(2)}(x)$ in the direction toward the oscillator. Small distance here means a distance smaller than that between the oscillator and the zero of $N_{N+1,p}^{(2)}(x)$ closest to the oscillator.

hence, we may calculate exactly the response of a detector at any location. Figure 2 indicates the calculated responses $r_i(j\omega)$ of detectors located at $x_1 = 2a/5$, $x_2 = 3a/5$, and $x_3 = 4a/5$. The locations of these detectors correspond to the zeros of the fifth spatial harmonic which is the lowest order, non-zero harmonic in $e_{3i}(j\omega)$. The constant c_{4p} is zero because of the location of the oscillator.

If the readings shown in Fig. 2 are analyzed according to Eq. (15), the three coefficients $A_{mp}(j\omega)$ depicted in Fig. 3 are inferred. Fitting these $A_{mp}(j\omega)$'s, by relations of the form given by Eq. (9), results in three ω_{mp} 's which are listed in Table I. Note that good results are obtained for the first two eigenvalues. It is seen from Fig. 3, however, that the phase of $A_{1p}(j\omega)$ at high fre-



Fig. 2. Magnitude and phase of the oscillating portion of the reading of i'th detector in a numerical oscillation test. Magnitude is normalized with respect to the steady-state reading of the detector.



Fig. 3. Comparison of exact behavior of expansion coefficients with that inferred from the results shown in Fig. 2. Detectors are located at zeros of the lowest order, non-negligible mode, not included in the NMA.

quencies is below its proper asymptotic value of -90°. To alleviate this error, the rule of thumb is used and all detectors are moved towards the oscillator by a distance of 10 cm. Repetition of the preceding analysis shows that the asymptotic phase of the new $A_{1p}(j\omega)$ is indeed closer to -90° and that the inferred eigenvalues are as shown in the last column of Table I. These estimates of the ω_{mp} 's are in excellent agreement with the known theoretical values.

Foulke's thesis⁶ contains additional results on numerical oscillation tests performed with both uniform and nonuniform reactors.

Experimental data from oscillation tests performed on the NORA reactor¹³ are shown in Fig. 4. If the readings of detectors B, D, and E are

¹³P. T. HANSSON and L. R. FOULKE, "Investigations in Spatial Reactor Kinetics," KR-43, NORA 3, Inst. of Atomenergi, Kjeller, Norway (1963). See also: *Nucl. Sci. Eng.*, **17**, 528 (1963).



on the NORA¹³ reactor.

analyzed according to Eq. (15), three frequencydependent modal expansion coefficients are deduced and they are shown in Figs. 5 and 6. Note that the coefficient of the first (fundamental) spatial harmonic is defined to include also the 14



Fig. 5. Comparison of theoretical magnitude of Fig. 4. Experimental data from the oscillation test expansion coefficients with that inferred from the data shown in Fig. 4.



Oscillation frequency, ω , rad /sec

Fig. 6. Comparison of theoretical phase of expansion coefficients with that inferred from the data shown in Fig. 4.

delayed-neutron precursor contributions, that is,

$$A_{1}(j\omega) = A_{1p}(j\omega) + \sum_{k=1}^{14} A_{1k}(j\omega) \quad . \tag{18}$$

This is done here because the first prompt thermal-neutron eigenvalue is of the same order of magnitude as the delayed-neutron eigenvalues. Superimposed on Figs. 5 and 6 are the theoretical results for $A_1(j\omega)$, $A_{2p}(j\omega)$, and $A_{3p}(j\omega)$. The natural modes used in this analysis are calculated by means of the computer code MUDMO-II,⁶ which was written for the purposes of this work.

The NORA experiments were not designed to measure the ω_{mp} 's of higher spatial harmonics. For this reason, the maximum frequency of oscillation is not high enough to allow estimation of the ω_{mp} 's. Nevertheless, it is seen from Figs. 5 and 6 that the inferred behavior of the expansion coefficients is reasonable. It should be noted that use of other expansion modes, such as Bessel functions for example, would require a much larger number of terms for the representation of the same experimental data on the NORA reactor.

Interpretation of Measurements for Subcriticality and Stability.

When spatial dynamic effects are important, the interpretation of some standard kinetic experiments may be ambiguous and the results may be grossly in error if only a single neutron detector is used. The error may be reduced through use of more than one detector and through analysis of the detector readings by means of a natural mode expansion.

Subcriticality

A measure of subcriticality of the fundamental spatial harmonic is given by ω_{1p} which is often inferred from an oscillation test. To see the errors which may arise in such a test, consider the following numerical experiment. A uniform slab reactor of extrapolated width a = 240 cm is made subcritical by a uniform adjustment of the thermal-absorption cross section. The nuclear constants of this reactor are given in Table II.

Figure 7 shows the computed responses $r_i(j\omega)$ of three plane, thermal-neutron detectors located at $x_1 = a/4$, $x_2 = a/2$, and $x_3 = 3a/4$, induced by a plane, oscillating, thermal-neutron absorber at $x_0 = a/6$. These $r_i(j\omega)$'s are found from a closedform solution of the space- and frequency-dependent neutron density. Included in the figure are also the eigenvalues ω_{1p} derived either from the quasi-break frequency of each $r_i(j\omega)$ individually, or from the break frequency of the

$$A_{1p}(j\omega) \left[= c_{1p}(j\omega - \omega_{1p})^{-1} \right]$$

TABLE II

Information Concerning the 240-cm Slab Reactor Considered in the Subcriticality and Stability Examples

Nuclear Constants						
width = 240 cm $\Sigma_{a20} = 0.2661376 \text{ cm}^{-1}$ $\nu \Sigma_{f^2} = 0.497857 \text{ cm}^{-1}$ $D_2 = 0.409718 \text{ cm}$ $v_2 = 2.2 \times 10^5 \text{ cm/sec}$ $\beta = 0.0064$	$\Sigma_r = 0.0164444 \text{ cm}^{-1}$ $\Sigma_{a10} = 0.033515 \text{ cm}^{-1}$ $\nu \Sigma_{f1} = 0.0194962 \text{ cm}^{-1}$ $D_1 = 1.69531 \text{ cm}$ $v_1 = 4.06 \times 10^6 \text{ cm/sec}$ $\lambda = 0.08 \text{ sec}$					
For Subcriticality Example	: $\Sigma_{a2} = 0.2678515 \text{ cm}^{-1}$					
For Stability Example: $\gamma_2 v_2 N_0^{(2)}(x) = 2.55 \times 10^{10} (\text{cm}^3 \text{ sec }^\circ\text{C})^{-1}$ $\gamma_1 v_1 N_0^{(1)}(x) = 5.25 \times 10^8 (\text{cm}^3 \text{ sec}^\circ\text{C})^{-1}$ $g = 9.42 \times 10^{-12} \text{ cm}^3 - ^\circ\text{C}/\text{fission}, \tau_1 = 0.244 \text{ sec},$ $\tau_2 = 0.73 \text{ sec}, \tau_3 = 0.21 \text{ sec}.$						



Fig. 7. Comparison of actual magnitude of fundamental mode coefficient with that indicated either by individual detector readings or that inferred from all detector readings. Oscillation test on a subcritical reactor.

which results from the analysis of all the $r_i(j\omega)$'s simultaneously according to Eq. (15). These results indicate the error that may be introduced by spatial effects in subcriticality measurements and the reduction of the error through the use of a natural mode expansion. The natural modes in this simple example are sinusoids.

Stability

Small amplitude oscillation tests are often performed on reactors to estimate the power range for which the reactor is asymptotically stable.¹⁴ Attention must be paid to spatial effects to avoid erroneous interpretations of the results. To illustrate this point, consider the reactor model of the preceding example and assume that feedback is introduced via the group absorption cross sections. Specifically,

$$v_{1}\Sigma_{a1}N^{(1)}(x,t) = v_{1}\Sigma_{a10}N^{(1)}(x,t) + \gamma_{1}v_{1}N^{(1)}_{0}(x)\theta_{1}(x,t),$$
(19)

$$v_{2}\Sigma_{a2}N^{(2)}(x,t) = v_{2}\Sigma_{a20}N^{(2)}(x,t) + \gamma_{2}v_{2}N^{(2)}_{0}(x)\theta_{2}(x,t),$$
(20)

where

$$\theta_i(x,t) = T^{(i)}(x,t) - T_0^{(i)}(x)$$
 for $i = 1, 2$.

The two incremental temperatures satisfy the relations

$$\frac{\partial \theta_{1}(x,t)}{\partial t} = gv_{1} \Sigma_{f1} N^{(1)}(x,t) + gv_{2} \Sigma_{f2} N^{(2)}(x,t) - \frac{1}{\tau_{1}} \left[\theta_{1}(x,t) - \theta_{2}(x,t) \right]$$
(21)

and

$$\frac{\partial \theta_2(x,t)}{\partial t} = \frac{1}{\tau_2} \left[\theta_1(x,t) - \theta_2(x,t) \right] - \frac{1}{\tau_3} \theta_2(x,t) \quad .$$

The values of the feedback parameters are listed in Table II.

Figure 8 shows the computed responses $r_i(j\omega)$ of three plane, thermal-neutron detectors located at $x_1 = a/4$, $x_2 = a/2$, and $x_3 = 3a/4$, induced by a plane, oscillating, thermal-neutron absorber at $x_0 = a/6$. These $r_i(j\omega)$'s are found from a closed-form solution of the space- and frequency-dependent neutron density. Note in the figure that each result depends upon the location of the detector.

The useful quantity for asymptotic stability analysis is the frequency-dependent coefficient of the fundamental spatial harmonic. This coefficient can be inferred from the measured $r_i(j\omega)$'s by means of an analysis in terms of a natural mode expansion and it is shown in Fig. 8. Note the substantial phase difference between the phase of each of the $r_i(j\omega)$'s and that of the coefficient of the fundamental mode. The natural modes used in the analysis of this simple example are sinusoids.

III.2. Correlation Between Flux Tilting and Prompt Thermal-Neutron Eigenvalues

The susceptibility of a thermal reactor to flux tilting is a function of the prompt thermal-neutron eigenvalues. To see this clearly, consider the



Fig. 8. Magnitude and phase of fundamental mode coefficient inferred from three detector readings. Oscillation test on a reactor at power.

following example reactors whose properties are described in Table III. Reactor I is typical of a small, uniform, light-water-moderated assembly. Reactor II is typical of a large, uniform, lightwater-moderated assembly. Reactor III^d is similar to Reactor II except for "decoupling" which is introduced through the addition and removal of absorbing material in the central and from the outer regions, respectively.

Figure 9 shows the eigenvalue spectra of these three reactors. Note that although the prompt thermal-neutron eigenvalue ω_{1p} of the first spatial harmonic (β/Λ of point kinetics) is essentially the

¹⁴E. P. GYFTOPOULOS, "Theoretical and Experimental Criteria for Nonlinear Reactor Stability," *Nucl. Sci. Eng.*, **26**, 26 (1966).

^dReactor III is virtually identical to the 240-cm slab reactor described in Refs. 15 and 16.

¹⁵E. P. GYFTOPOULOS, "Point Reactor Kinetics and Stability Criteria," *Proc. Intern. Conf. Peaceful Uses At. Energy*, P/270, Vol. 4, 3 (1964).

¹⁶J. B. YASINSKY and A. F. HENRY, "Some Numerical Experiments Concerning Space-Time Reactor Kinetics Behavior," *Nucl. Sci. Eng.*, **22**, 171 (1965).

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TABLE III					
Nuclear	Constants of Reactors	I.	II.	and	II

			Reactor III			
Parameter	Reactor I	Reactor II	Region 1	Region 2	Region 3	
width, cm Σ_{a1} , cm ⁻¹ Σ_{a2} , cm ⁻¹ $\nu \Sigma_{f1}$, cm ⁻¹ $\nu \Sigma_{f2}$, cm ⁻¹ Σ_{τ} , cm ⁻¹ D_{1} , cm D_{2} , cm ν_{1} , cm/sec ν_{2} , cm/sec	$\begin{array}{c} 60.\\ 0.033706\\ 0.266184\\ 0.0213143\\ 0.544286\\ 0.0164444\\ 1.69531\\ 0.409718\\ 4.06\times10^{6}\\ 2.2\ \times10^{5} \end{array}$	$\begin{array}{c} 240,\\ 0.033515\\ 0.2661376\\ 0.0194962\\ 0.497857\\ 0.0164444\\ 1.69531\\ 0.409718\\ 4.06\times10^6\\ 2.2\times10^5 \end{array}$	60. 0.0322365 0.2658284	$\begin{array}{c} 120.\\ 0.0340843\\ 0.2662750\\ 0.0194962\\ 0.497857\\ 0.0164444\\ 1.69531\\ 0.409718\\ 4.06\times10^{6}\\ 2.2\ \times10^{5}\\ \end{array}$	60. 0.0322026 0.2658203	
β λ, sec^{-1}	0.0064 0.08	0.0064 0.08	0.0064 0.08			



Fig. 9. Eigenvalue spectra of Reactors I, II, and III.

same for all three reactors, the ω_{mp} 's for m > 1 are considerably different. In particular, note that the ratio ω_{1p}/ω_{2p} is largest for Reactor III and smallest for Reactor I.

Many calculations of transients following step perturbations applied to each of Reactors I, II, and III were performed by means of the NMA. The perturbation in each calculation is a step removal of thermal-neutron absorber from the same fractional, localized, reactor region. A tilt parameter ϵ is defined by the relation

$$\epsilon \equiv (\omega_{1p} / \omega_{2p})(PP) , \qquad (23)$$

where (PP) is the perturbation parameter given by the relation

$$(PP) \equiv \frac{1}{\omega_{1p}} \frac{\langle N_0^{*(2)}(x) \ v_2 \delta \Sigma_{a2}(x) \ N_0^{(2)}(x) \rangle}{\langle N_0^{*(1)}(x) \ N_0^{(1)}(x) + N_0^{*(2)}(x) \ N_0^{(2)}(x) \rangle},$$
(24)

and $\delta \Sigma_{a2}(x)$ is the step change in the thermal-

neutron cross section. The neutron group densities $N_0^{(1)}(x)$ and $N_0^{(2)}(x)$, and the adjoint densities $N_0^{*(1)}(x)$ and $N_0^{*(2)}(x)$ are the unperturbed distributions. The perturbation parameter equals the reactivity (in dollar units) that would be calculated for a fundamental-mode, space-independent kinetics approximation.

A tilt index $\tau(t)$ is defined by the relation

$$\tau(t) = \frac{\sum_{k=1}^{K} A_{2k}(t)}{\sum_{k=1}^{K} A_{1k}(t)} , \qquad (25)$$

with the conditions that the thermal-neutron components of $\psi_{1k}(x)$ and $\psi_{2k}(x)$ are normalized so that

$$N_{1k}^{(2)}(a/2) = N_{2k}^{(2)}(a/4) = 1.0 \text{ for } k = 1, 2, \dots K$$
 .
(26)

The asymptotic tilt index τ_a equals $\tau(\infty)$, and it is a measure of the amplitude of the second spatial harmonic over that of the first (fundamental) spatial harmonic when the asymptotic flux shape corresponding to the step perturbation is achieved.

The results of the step transients mentioned above are shown in Fig. 10. They are given in terms of the asymptotic tilt index τ_a vs the tilt parameter ϵ . The perturbations are introduced over the region $0 \le x \le a/4$, and five modes (M=5)are retained in the NMA. It is seen from this figure that the susceptibility of these example reactors to flux tilting, as measured by the index τ_a , correlates rather well with ϵ . It follows, therefore, that, for localized perturbations which



Fig. 10. Correlation between asymptotic tilt index and tilt parameter.

would be equivalent from the point of view of perturbation theory, the larger the ratio ω_{1p}/ω_{2p} , the larger the flux tilting.

III.3. Calculation of Step Transients With and Without Feedback

The first series of calculations is for the purpose of illustrating how the number of modes, required for an accurate representation of transients in reactors without feedback, depends on: 1) the magnitude of the perturbation parameter (*PP*): and 2) the degree of localization of the step change in the absorption cross section. Two perturbations are considered for Reactor II. Perturbation I is applied in the region $0 \le x \le a/4$, and it is representative of changes with a low degree of localization. Perturbation II is applied in the region $a/8 \le x \le a/4$, and it is representative of changes with a high degree of localization. The "exact" solution in each case is determined numerically by means of the computer code MUDMO-II.6

Figure 11 illustrates the error in the calculated, reciprocal asymptotic period as a function of the number M of spatial harmonics retained in the NMA when Perturbation I is analyzed. The magnitude of the perturbation is changed so that (PP) varies from 0.25 to 1.00. Figure 11a shows the error vs M when epithermal modes are neglected. Figure 11b shows the error vs M when epithermal modes are included. It is seen from these figures that the reciprocal asymptotic period is estimated to within ten percent in all cases when five spatial harmonics are retained. The inclusion of epithermal modes decreases the error by about two percent. Finally, note that the convergence to the exact period is best for the smallest perturbation (PP) = 0.25, and worst for the intermediate perturbation (PP) = 0.50. From many results of this type we conclude that the NMA is particularly suitable either for slow transients with periods of the order of seconds or for fast transients with periods of the order of milliseconds or less.

Figure 12 is similar to Fig. 11 except that here Perturbation II is analyzed. Epithermal modes are included and the magnitude of the perturbation corresponds to (PP) = 0.75. It is seen from Fig. 12 that at least eight spatial harmonics must be retained to estimate the reciprocal asymptotic period to within ten percent of its exact value.

From the results of Figs. 11 and 12 we conclude that for an accurate representation of a step transient initiated by a perturbation of width Δx , at least M + 1 harmonics must be retained in the NMA, where M is the spatial harmonic whose wavelength is less than $2\Delta x$.

The second series of calculations illustrates



Fig. 11. Percent error in the estimated asymptotic period as a function of the order of the NMA. Data correspond to Perturbation I applied to Reactor II.

the application of the NMA to the analysis of a power excursion in a reactor model with an inherent shutdown mechanism.

A three-region slab reactor, 60-cm thick, is considered. The nuclear data are given in Table IV. The power excursion is initiated by a step increase of the fission cross section in Region 1, $0 \le x \le 15$ cm. Shutdown is introduced by the change of the thermal absorption cross section of each region in proportion to the energy released in the same region. Specifically,

$$\Sigma_{a2}^{(i)} = \Sigma_{a20}^{(i)} + \gamma_i E_i(t) \quad \text{for } i = 1, 2, 3 \quad , \qquad (27)$$

$$E_{i}(t) = v_{2} \int_{\text{region } i} dx' \int_{0}^{t} dt' \ N^{(2)}(x', t') \quad , \qquad (28)$$

and γ_i is the feedback coefficient of the *i*'th region.

The dynamic equations that describe this reactor model are given by the matrix relation



Fig. 12. Percent error in the estimated asymptotic period as a function of the order of the NMA. Data correspond to Perturbation II applied to Reactor II.

$$\frac{\partial \psi(x,t)}{\partial t} = [H_0]\psi(x,t) + [h]\psi(x,t) + f \quad , \qquad (29)$$

where

$$\begin{bmatrix} H_0 \end{bmatrix} = \begin{bmatrix} v_1 \frac{\partial}{\partial x} D_1 \frac{\partial}{\partial x} - \alpha_1 & (1-\beta) v_2 v \Sigma_{f2} & \lambda \\ v_1 \Sigma_r & v_2 \frac{\partial}{\partial x} D_2 \frac{\partial}{\partial x} - \alpha_2 & 0 \\ \beta v_1 v \Sigma_{f1} & \beta v_2 v \Sigma_{f2} & -\lambda \end{bmatrix}$$

$$\alpha_1 = v_1 \begin{bmatrix} \Sigma_{a1} + \Sigma_r - (1-\beta) v \Sigma_{f1} \end{bmatrix}, \quad \alpha_2 = v_2 \begin{bmatrix} \Sigma_{a2} \end{bmatrix},$$

$$\begin{bmatrix} (1-\beta) v_1 v \delta \Sigma_{f1} & (1-\beta) v_2 v \delta \Sigma_{f2} & 0 \\ 0 & 0 & 0 \\ \beta v_1 v \delta \Sigma_{f1} & \beta v_2 v \delta \Sigma_{f2} & 0 \end{bmatrix},$$

$$\Psi(x,t) = \operatorname{col} \begin{bmatrix} N^{(1)}(x,t) & N^{(2)}(x,t) & C(x,t) \end{bmatrix},$$

$$f = \operatorname{col} \begin{bmatrix} 0, -v_2 \sum_{i=1}^3 \Delta_i \gamma_i E_i(t) N^{(2)}(x,t) & 0 \end{bmatrix},$$
(30)

and $\Delta_i = 1$ for x in Region i, and $\Delta_i = 0$ otherwise.

An approximate solution of Eq. (29) is found by expanding $\Psi(x,t)$ in terms of the eigenvectors of the operator $[H_0]$, that is,

$$\boldsymbol{\psi}(x,t) \simeq \sum_{m=1}^{M} \sum_{k=1}^{3} A_{mk}(t) \boldsymbol{\psi}_{mk}(x) \quad , \qquad (31)$$

where

$$[H_0] \boldsymbol{\psi}_{mk} = \omega_{mk} \boldsymbol{\psi}_{mk}$$
 .

The substitute, adjoint weight, and integrate procedure outlined in Sec. II leads to a set of 3M

TABLE IV

Nuclear Constants of the Reactor System Considered in the Power Excursion Calculation

Parameter	Region 1	Region 2	Region 3		
width, cm	0 to 15 cm	15 to 45 cm	45 to 60 cm		
D_1 , cm D_2 , cm Σ_r , cm ⁻¹	1.69531 0.409718 0.016444				
$ \begin{array}{c} \Sigma_{a1}, \ \mathrm{cm}^{-1} \\ \Sigma_{a2}, \ \mathrm{cm}^{-1} \\ \nu \ \Sigma_{f1}, \ \mathrm{cm}^{-1} \ (t < 0) \\ \nu \ \Sigma_{f1}, \ \mathrm{cm}^{-1} \ (t \ge 0) \\ \nu \ \Sigma_{f2}, \ \mathrm{cm}^{-1} \ (t \ge 0) \\ \nu \ \Sigma_{f2}, \ \mathrm{cm}^{-1} \ (t \ge 0) \\ \gamma_{f} \end{array} $	$\begin{array}{c} 0.0322302\\ 0.265827\\ 0.0213296\\ 0.025733\\ 0.544678\\ 0.65712\\ 6.515\times10^{-6} \end{array}$	$\begin{array}{c} 0.0340950\\ 0.266278\\ 0.0213296\\ 0.0213296\\ 0.544678\\ 0.544678\\ 2.7\times10^{-6}\\ \end{array}$	$\begin{array}{c} 0.0322302\\ 0.265827\\ 0.0213296\\ 0.0213296\\ 0.544678\\ 0.544678\\ 5.4\times10^{-6}\\ \end{array}$		

nonlinear, ordinary differential equations which may be solved numerically for the coefficients $A_{mk}(t)$. By considering the solution in different time ranges, the system of 3M equations can be effectively reduced to three separate problems, each involving only M equations. This reduction is possible because of the fact that the eigenvalues ω_{mk} fall into three distinct clusters of different magnitudes.

Results obtained for M = 3 and M = 4 are shown in Fig. 13. Superimposed on the same figure are "exact"¹⁷ results derived by means of a direct numerical solution of Eq. (29). It is seen from this figure that the NMA solution for M = 4 is in good agreement with the "exact" results. The calculational advantages of the NMA are that: 1) less computer time is required if many problems are to be solved for the same reactor; and 2) it may be more readily extendable to multidimensional systems.

The accuracy of the NMA results for power excursions can be improved without appreciable computational effort through use of "correction modes." Correction modes are approximations to spatial harmonics beyond the M'th and are introduced in an algebraic manner, i.e., in a manner that does not introduce additional differential equations. To clarify this idea, consider the expansion of the solution vector of Eq. (29) to be of the form

$$\psi(x,t) = \sum_{m=1}^{M} \sum_{k=1}^{3} A_{mk}(t) \psi_{mk}(x) + \sum_{m=M+1}^{Q} \sum_{k=1}^{3} A_{mk}(t) \psi_{mk}(x) . \qquad (32)$$



Fig. 13. Volume integral of thermal-neutron density vs time, for a power excursion calculation.

This form involves 3(Q - M) more coefficients than Eq. (31). If the additional 3(Q - M) coefficients were to be computed in the usual manner of the NMA there would be no savings in computational effort. The following simplifications, however, seem to be reasonable for certain problems: 1) spatial harmonics beyond the M'th do not make appreciable nonlinear contributions because their amplitude is small; and 2) the prompt jump approximation $dA_{mk}(t)/dt = 0$ can be made for the prompt thermal-neutron modes when $|\omega_{mp}|$ for m > M is large compared with the inverse, initial asymptotic period. These simplifications are extremely attractive if the natural modes are easy to find for spatial harmonics beyond the M'th.

For the reactor model described by the properties tabulated in Table IV it is noted that, as mincreases, the spatial shapes of the natural modes approach those of an equivalent uniform reactor (i.e., sinusoids). In addition, it is noted that the spectra of the prompt thermal-neutron eigenvalues, for the nonuniform reactor and its uniform version, become quite similar as m increases. Hence, the correction modes for this example may be taken as the natural modes of the equivalent uniform reactor. Results obtained using correction modes for M = 3 and M = 4 are shown in Fig. 13. In each case, Q = 8. Note that the results for both M = 3 and M = 4 agree very well with the

¹⁷M. RADD, "WIGLE-40, A Two-Group, Time-Dependent Diffusion Theory Program for the IBM 7040 Computer," IDO-17125, USAEC, Idaho (1965).

"exact" results. A summary of results using different degrees of approximation is given in Table V.

Finally, it should be noted that the computer time used by Wigle¹⁷ with six groups of delayed neutrons for the calculation from 0 to 18 msec was 4.8 min. The NMA results for M = 4 with four correction modes required 4.2 min for a calculation from 0 to 15 msec. Of this time, 2 min were consumed for the calculation of the natural modes, 0.5 min for the calculation of the necessary integrals appearing in the NMA, and 1.7 min for the integration of the resultant ordinary differential equations. Subsequent calculations on the same reactor would require only the time necessary for the integration of the differential equations.

TABLE V

Comparison of Calculations of the Asymptotic Period and the Power Using Different Approximations

Calculation	Initial Asymptotic Period <i>T</i> , msec	Power at t = 10 msec (Arbitrary Units)
WIGLE ¹⁷	0,504	3.77
Natural Mode Approximation:		
Without Correction Modes M = 3	0.532	1.70
Without Correction Modes M = 4	0.509	3.15
With Correction Modes M = 3	0.505	3.55
With Correction Modes M = 4	0.505	3.55

IV. CALCULATION OF TWO-DIMENSIONAL MODES

The purpose of this section is to present preliminary results of calculations of higher order, two-dimensional natural modes by means of a "synthesis"⁵ technique used in conjunction with a "Stabilized March Technique"¹⁸. The synthesis technique consists effectively of reducing an xand z-dependent problem to a z-dependent problem by expanding each x- and z-dependent mode in a series of products of unknown z-dependent coefficients, and known x-dependent trial functions. For example, expand the g'th component of the prompt thermal-neutron mode of the (m, n)'th spatial harmonic (m'th harmonic in the *x*-direction and *n*'th harmonic in the *z*-direction) as

$$N_{mn,p}^{(g)}(x,z) \cong \sum_{j=1}^{J} X_{mp}^{(g,j)}(x) Z_{np}^{(g,j)}(z) .$$
(33)

The substitute, adjoint weight, and integrate procedure then leads to a set of differential equations which may be solved for $Z_{np}^{(g,j)}(z)$ by a numerical method such as the Stabilized March Technique. A pair of computer codes, SYNSIG and MUDMO-II, developed for the construction of modes in this manner, is described by Foulke.⁶

The preceding technique is used to find natural modes of the two-dimensional reactor example shown in Fig. 14. The nuclear data for this example are given in Table VI. Figures 15 and 16



Fig. 14. Reactor geometry for two-dimensional mode calculations.

TABLE VI

Nuclear Parameters of a Two-Dimensional Reactor

Parameter	Region 1	Region 2	Region 3
$\sum_{a1}, \text{ cm}^{-1}$ $\sum_{a2}, \text{ cm}^{-1}$	0.0322365 0.265828	0.0340843 0.266275	0.0322026 0.265820
$\nu \Sigma_{/1}, \text{ cm}^{-1}$ $\nu \Sigma_{/2}, \text{ cm}^{-1}$ $\Sigma_{r}, \text{ cm}^{-1}$ $D_{1}, \text{ cm}$ $D_{2}, \text{ cm}$ $\nu_{1}, \text{ cm/sec}$ $\nu_{2}, \text{ cm/sec}$ β $\lambda, \text{ sec}^{-1}$		$\begin{array}{c} 0.0194903\\ 0.497707\\ 0.0164444\\ 1.69531\\ 0.409718\\ 4.06\times10^6\\ 2.2\times10^5\\ 0.0064\\ 0.08\\ \end{array}$	

¹⁸D. R. EDWARDS and K. F. HANSEN, "The Stabilized March Technique Applied to the Diffusion Equation," *Nucl. Sci. Eng.*, **25**, 58 (1966).



Fig. 15. Synthesized thermal-neutron component of the two-dimensional natural mode $\psi_{11,p}(x,z)$.

show the thermal-neutron components of the eigenvectors $\Psi_{11,p}(x,z)$ and $\Psi_{22,p}(x,z)$, respectively. The spectrum of the prompt thermal-neutron eigenvalues $\omega_{mn,p}$ is given in Table VII.

Since there are no exact calculations of twodimensional modes, the success of the procedure outlined above may be judged by considering how well the orthogonality relations

$$\langle \psi_{mn,p}^{*}(x,z), \psi_{rs,p}(x,z) \rangle = \delta_{mr} \delta_{ns}$$
 (34)

are satisfied. Table VIII presents the results of testing the orthogonality relations for the example under consideration. Note that in some cases the

TABLE VII

Spectrum of the Prompt Thermal-Neutron Eigenvalues of the Natural Modes of a Two-Dimensional Reactor

	x index, m						
z index, n	1	2	3	4			
1	-416.6	-1036.3	-2829.3	-5681.0			
2	-1049.1	-1869.8	(-3634.) ^a	-6129.8			
3	-2592.4	-3603.3	-5391.2	-7804.3			
4	-5656.2	-5829.2	-7834.8	~10126.			

^aCoincided with extraneous eigenvalue, could not calculate eigenvector. Changed weighting vector from adjoint eigenvector to unit vector to break multiplicity; eigenvalue was then -3670.



Fig. 16. Synthesized thermal-neutron component of the two-dimensional natural mode $\Psi_{22,p}(x,z)$.

modes are not orthogonal. This means that the trial functions used to construct either the mode or its adjoint are not suitable. More experience with the behavior of two-dimensional modes will overcome this difficulty.

TABLE VIII

Results of a Check on the Orthogonality of a Number of the Synthesized Natural Modes of a Two-Dimensional Reactor Normalized Value of $\langle \Psi *_{m_{f}p} (x,z), \Psi_{ns,p} (x,z) \rangle$

n,s m,r	1,1	1,2	1,3	1,4	2,1	2,2	2,3	2,4
1,1	1.0	+	+	+	+	+	+	+
1,2	+	1.0	+	+	+	+	+	0.083
1,3	+	+	1.0	+	0.011	+	+	+
1,4	+	+	+	1.0	+	+	+	+
2,1	+	+	0.217	+	1.0	+	+	+
2,2	+	+	+	-0.034	+	1.0	+	+
2,3	-0.031	+	+	+	+	+	1.0	+
2,4	+	-0.096	+	+	+	+	+	1.0

+ Denotes that the magnitude is less than 0.01.

Another difficulty involved with the construction of two-dimensional natural modes is that the synthesis technique introduces extraneous eigenvalues. The synthesis technique transforms a two-dimensional problem with K-dependent variables into an approximate, one-dimensional problem with $J \cdot K$ -dependent variables, where J is the number of trial functions used in Eq. (33). Therefore, the approximate, one-dimensional problem has $J \cdot K$ eigenvalues associated with the (m,n)'th spatial harmonic, (J-1)K of which are extraneous. This difficulty can be overcome. For example, MUDMO-II can be used to scan the entire range of eigenvalue magnitudes so as to locate the eigenvalues roughly. After a few of the modes are determined, a pattern of relative behavior is recognized which allows the actual and extraneous eigenvalues to be separated. Difficulties caused by extraneous eigenvalues can be quite serious if an actual and an extraneous eigenvalue are so close in magnitude that they appear as a multiple eigenvalue. MUDMO-II cannot calculate the eigenvectors for multiple eigenvalues. This situation occurred in the attempt to synthesize $\psi_{32,p}(x,z)$ for the reactor shown in Fig. 14. The multiplicity was broken, however, by a change in the weighting function $X_{3p}^{*(g,j)}(x)$.

V. CONCLUSIONS

The NMA has been shown to be suitable for a broad class of kinetic problems and to involve experimentally verifiable parameters. An experimental procedure is outlined for the verification of the parameters by means of small signal oscillation tests. It is shown that the relative magnitudes of the verifiable parameters are sensitive functions of the geometrical arrangement of the reactor and are sensitive indicators of the susceptibility of the neutron density to undergo shape changes following a localized perturbation. The same experimental procedure is also used to interpret subcriticality and stability measurements when the experimental observations are functions of position and time.

A series of calculations of transients in onedimensional reactor models without feedback indicates that the number of spatial harmonics required for accurate representation of a transient is a function of the magnitude of the perturbation, and the degree of localization of the perturbation.

A power excursion in a reactor model with feedback is calculated with a low-order NMA. The results compare well with those of independent calculations which are considered to be exact. The introduction of correction modes gives an increase in accuracy of a low-order NMA without a corresponding increase in computational effort.

One of the major difficulties associated with the NMA, namely, the calculation of multidimensional natural modes, is attacked by means of a synthesis technique. The preliminary results indicate that such calculations are feasible.

APPENDIX A

Consider a reactor at very low power described by the equation

$$[H'_0] \psi'_0(x) + S_0(x) = 0 . \qquad (A.1)$$

The vector $\boldsymbol{\psi}_{0}'(\boldsymbol{x})$ is a *K*-vector containing the *K*-dependent variables of the system as components; $[H_{0}']$ is a $K \times K$ matrix operator governing the relationships between variables; and $\boldsymbol{S}_{0}(\boldsymbol{x})$ is a vector containing external sources. Let the reactor be excited by a localized, thermal-neutron absorber which oscillates sinusoidally with frequency ω . The kinetic behavior of the perturbed system is described by the equation

$$[H_0(\boldsymbol{x})] \boldsymbol{\psi}(\boldsymbol{x},t) + [h] \boldsymbol{\psi}(\boldsymbol{x},t) \exp(j\omega t)$$

+ $\boldsymbol{S}_0(\boldsymbol{x}) = \frac{\partial \boldsymbol{\psi}(\boldsymbol{x},t)}{\partial t}$, (A.2)

where it is to be understood that only the real parts of complex numbers will be considered. The perturbation matrix operator [h] contains only one element which gives the magnitude and location of the oscillating thermal-neutron absorber.

After the perturbation is introduced, the mean value $\psi_0(\mathbf{x})$ of $\psi(\mathbf{x},t)$ will be different from $\psi'_0(\mathbf{x})$ of Eq. (A.1). In addition, the operator $[H_0]$ will be slightly different from $[H'_0]$ of Eq. (A.1). One reason for the differences is that the introduction of the perturbation mechanism must be accompanied by a small change in reactor properties to keep the reactor critical. Another reason¹⁹ for the differences is that the product of $[h] \exp(j\omega t)$ and $[\psi(\mathbf{x}, t) - \psi_0(\mathbf{x})]$ causes the power level to increase. This increase, which must be balanced by a change in $[H'_0]$, is neglected. It is assumed that the variation of $\Psi(\mathbf{x},t)$ from its mean value $\Psi_0(\mathbf{x})$, is so small that $[h] \psi(\mathbf{x}, t)$ may be replaced by $[h] \psi_0(x)$. Now, if it is assumed that the solution vector may be expanded in a finite series of the natural modes of $[H_0]$ as

$$\boldsymbol{\psi}(\boldsymbol{x},t) \simeq \boldsymbol{\psi}_{0}(\boldsymbol{x})$$

$$+ \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(j\omega) \exp(j\omega t) \boldsymbol{\psi}_{mk}(\boldsymbol{x}) ,$$

¹⁹A. A. WASSERMAN, "Contributions to Two Problems in Space-Independent Nuclear Reactor Dynamics," IDO-16755, USAEC, Idaho (1962).

then the usual substitute, adjoint weight, and integrate procedure yields the following expression for the expansion coefficients:

$$A_{mk}(j\omega) = c_{mk} (j\omega - \omega_{mk})^{-1} \text{ for}$$
$$m = 1, 2, \dots, M, \text{ and}$$
$$k = 1, 2, \dots, K,$$

if it is assumed that

$$[H_0] \psi_0(x) + S_0(x) = 0 .$$

•

The constant c_{mk} is given by the relation

$$c_{mk} = rac{ < oldsymbol{\psi}_{mk}^{*}\left(oldsymbol{x}
ight), \left[h
ight]oldsymbol{\psi}_{0}\left(oldsymbol{x}
ight) > }{ < oldsymbol{\psi}_{mk}^{*}\left(oldsymbol{x}
ight), oldsymbol{\psi}_{mk}\left(oldsymbol{x}
ight) > }$$

The reading $R_i(t)$ of the *i*'th thermal-neutron detector is composed of an oscillating portion $r_i(j\omega) \exp(j\omega t)$, superimposed upon a time-averaged detector reading R_i^0 . The oscillating portion is given by

$$egin{aligned} &r_i\left(j\,\omega
ight)\,\exp(j\,\omega\,t)\,=\ &< v_2\,\sigma_i^{(2)}\left(oldsymbol{x}
ight)\,\sum_{m=1}^M\,\,\sum_{k=1}^K\,\,A_{mk}\left(j\,\omega
ight)\,\exp(j\,\omega\,t)\,N_{mk}^{(2)}\left(oldsymbol{x}
ight)>\ &=\,\,\sum_{m=1}^M\,\,\sum_{k=1}^K\,\,d_{mk}^{(i)}\,\,A_{mk}(j\,\omega)\,\exp(j\,\omega\,t)$$
 ,

where

$$d_{mk}^{(i)} \equiv \langle v_2 \sigma_i^{(2)}(\boldsymbol{x}) N_{mk}^{(2)}(\boldsymbol{x}) \rangle$$
.

The function $\sigma_i^{(2)}(\boldsymbol{x})$ is the response function of the *i*'th thermal-neutron detector. It depends upon the size and location of the detector.