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The Application of Topological Methods to the Kinetics of Homogeneous Reactors

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The step and ramp responses of homogeneous reactors with a single negative temperature coefficient of reactivity and delayed neutrons are studied by means of topological methods. The properties of stability in the small and in the large are derived for the first time and particular solutions are found. An approximate analytical solution is also proposed in the case of no heat losses (adiabatic reactor model) and small or large reactivity variation.

INTRODUCTION

In this paper, the kinetic behavior of a homogeneous nuclear reactor with one negative temperature coefficient of reactivity is investigated. It is assumed that the familiar space-independent kinetic equations (*1*) hold

$$\frac{dn}{dt} = \frac{k - \beta}{l} n + \sum \lambda_i C_i \qquad (1)$$

$$\frac{dC_i}{dt} = \frac{\beta_i n}{l} - \lambda_i C_i \qquad (2)$$

and that the reactor temperature is related to the neutron density by the differential equation

$$\frac{dT}{dt} = a(n - n_0^*) - gT \qquad (3)$$

The reactivity is a linear function of the temperature

$$k = k_0 - rT \qquad r > 0 \qquad (4)$$

Positive temperature coefficients of reactivity are not considered because then the nuclear reactor is completely unstable. In order to simplify the study, it is assumed, without loss of generality, that the effect of the delayed neutrons can be represented by one group. Here the response of the reactor described by Eqs. (1), (2), (3), and (4) is investigated for step and ramp variations of reactivity.

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Many particular cases of the present problem have been studied by various authors. In particular, J. Chernick (*2*) and W. K. Ergen *et al.* (*3*) considered the case of reactors without delayed neutrons ($\beta = 0$). The most important contribution is by R. S. Margulies (*4*), who applied the classical theory of differential equations to the present problem. Margulies' results deal mostly with the problem of linear stability. Some nonlinear aspects have been treated by W. K. Ergen (*5*), who proved that no periodic solutions exist for $g = 0$. The problem of boundedness has not been given a definite answer.

Here the analysis is performed by means of topological methods. These methods, well-known in the field of nonlinear mechanics (*6*, *7*, *8*), are very powerful and allow one to reach definite conclusions about the properties of the solution of nonlinear systems. They consist in a study of the behavior of the solutions of a set of differential equations in the phase space of the dependent variables. The phase space consists of equilibrium points and trajectories. An equilibrium point is a state in which the physical system remains, if not perturbed. A trajectory is a series of nonequilibrium states starting from an equilibrium point. A system is said to be unstable if a trajectory goes to infinity. It is asymptotically stable in the sense of Liapounoff if all trajectories end up at an equilibrium point.

In the study of the trajectories some very useful properties can be derived by means of the powerful Liapounoff's second method (*8*) when a Liapounoff function of the differential system can be found.

When successfully applied, topological methods give the following information:

(a) whether equilibrium points exist

(b) whether the solutions are damped or oscillatory

(c) whether periodic solutions exist, and

(d) whether all solutions are bounded.

In some cases, it is possible to find upper or lower bounds to the variables and to indicate the influence of certain parameters on the value of the extrema. Previously, some interesting aspects of the application of topological methods to nuclear reactors have been given by J. M. Stein (*9*), W. C. Sangren (*10*), J. Chernick (*2*), V. K. Paré (*11*), S. Visner *et al.* (*12*), T. A. Welton (*13*, *14*), W. K. Ergen (*5*), W. K. Ergen *et al.* (*3*, *15*) H. J. Lipkin (*16*), and H. J. Lipkin *et al.* (*17*).

In this paper, three models of a reactor are considered:

$n_0^* = 0$		(Newton's law of cooling model)
$n_0^* = g = 0$		(Adiabatic model)
$g = 0$	$n_0^* \neq 0$	(Constant power removal model)

The response to a step change of reactivity is first analyzed for the three models and then the response to a ramp is considered for the last two models. The present analysis can also be applied to problems in which the external coolant temperature is varied suddenly. In the following sections, the properties of stability in the small and in the large and the existence of particular solutions are investigated.

STEP RESPONSE—NEWTON'S LAW OF COOLING MODEL: $n_0^* = 0$

Newton's law of cooling model is a good representation of the heat exchange in homogeneous reactors in which the coolant temperature remains essentially constant. In order to reduce the complexity of the analysis it is assumed that the heat exchange coefficients are constant. All the conclusions derived in the following paragraph are in qualitative agreement with experiments.

Stability in the Small

If the reactor is described by Eqs. (1), (2), (3), and (4) with $n_0^* = 0$, there are only two equilibrium points O and X (Fig. 1). The coordinates are

$$O\begin{cases} n = 0 \\ C = 0 \\ T = 0 \end{cases} \qquad X\begin{cases} n = n_0 = gk_0/(ar) \\ C = C_0 = \beta g k_0/(arl\lambda) \\ T = T_0 = k_0/r \end{cases}$$

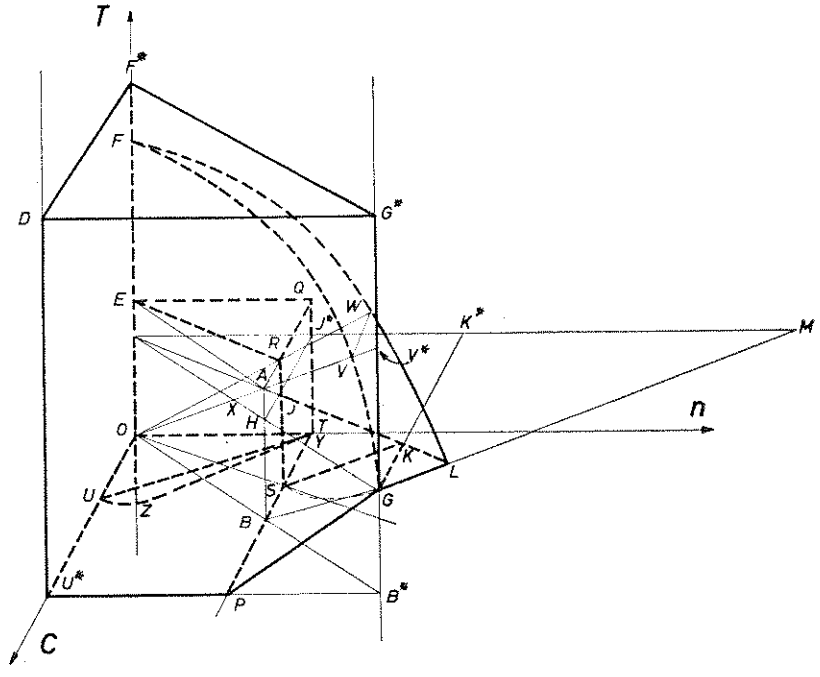

Fig. 1. Newton's law of cooling model. Phase space. Boundedness of all trajectories.

In view of the well-known Liapounoff's theorem, the behavior of the trajectories in the neighborhood of an equilibrium point can be studied by linearizing the differential equations, provided that the Jacobian of the differential system is not zero.

The linearized kinetic equations near O are

$$\begin{aligned} d\bar{n}/dt &= [(k_0 - \beta)/l]\bar{n} + \lambda\bar{C} \\ d\bar{C}/dt &= (\beta/l)\bar{n} - \lambda\bar{C} \\ d\bar{T}/dt &= a\bar{n} - g\bar{T} \end{aligned} \tag{5}$$

where the overbarred symbols mean the variation of n, C, and T around the equilibrium value. The transient behavior of the solutions near the origin O depends on the location in the complex plane of the characteristic roots of Eq. (5). They are solutions of

$$\begin{vmatrix} [(k_0 - \beta)/l] - s & \lambda & 0 \\ \beta/l & -\lambda - s & 0 \\ a & 0 & -g - s \end{vmatrix} = 0 \tag{6}$$

or

$$(s + g)\left[s^2 + \left(\lambda - \frac{k_0 - \beta}{l}\right)s - \frac{\lambda k_0}{l}\right] = 0 \tag{7}$$

If the reactor is subcritical at zero power, i.e., $k_0 < 0$, all the roots are real and negative and the solutions near O are damped exponentials. If the reactor is supercritical at zero power, i.e., $k_0 > 0$, one of the roots is positive. Hence the equilibrium point O is unstable. If the reactor is critical at zero power, i.e., $k_0 = 0$, the analysis of linearized equations (5) does not give an answer because the condition of applicability of Liapounoff's theorem (*18*) is notsatis fied. However, it can easily be shown that

$$\frac{d}{dt}(L_1) \equiv \frac{d}{dt}\left(n + C + \frac{r}{al}\frac{T^2}{2}\right) = \frac{-grT^2}{al} \tag{8}$$

if $k_0 = 0$.

L_1 is a positive definite function with a negative derivative [Liapounoff's function (*8*)]. Equation (8) shows that all trajectories tend toward the origin and therefore that the origin is asymptotically stable for $k_0 = 0$. In his study of the kinetics of nuclear reactors, R. S. Margulies (*4*) reaches the conclusion that in this case the reactor is "in general unstable" when it is actually always stable. Errors like this one are not uncommon when one tries to apply linear techniques to nonlinear problems.

Using the same method for the second equilibrium point, it is found that the characteristic roots s are solutions of

$$s^3 + \left(g + \lambda + \frac{\beta}{l}\right)s^2 + \left(\lambda g + g\frac{\beta}{l} + \frac{rn_0 a}{l}\right)s + \frac{a\lambda r n_0}{l} = 0 \tag{9}$$

Only the case $k_0 > 0$ is to be considered here because for $k_0 < 0$ the point X is in a region of no physical significance and X and O coalesce for $k_0 = 0$. If $k_0 > 0$, the second equilibrium point is always stable because the Hurwitz-Routh criterion (*19*) is satisfied for all values of n_0.

Writing Eq. (9) as

$$s^3 + b_1 s^2 + b_2 s + b_3 = 0 \tag{10}$$

and introducing the change of variable

$$y = s + (b_1/3) \tag{11}$$

Eq. (10) becomes

$$y^3 + py + q = 0 \tag{12}$$

The necessary and sufficient condition for all roots to be real is

$$(q^2/4) + (p^3/27) < 0 \tag{13}$$

and for two roots to be complex is

$$(q^2/4) + (p^3/27) > 0 \tag{14}$$

If the neutron lifetime is very small and $g \ll \lambda$, the condition for oscillatory behavior is

$$k_0/\beta > g/4\lambda$$

Stability in the Large

After studying the behavior of all solutions near the equilibrium points O and X, let us consider the problem of boundedness of all solutions.

If $k_0 \leqq 0$, the function L_1

$$L_1 = \left(n + C + \frac{r}{2al}T^2\right) \tag{15}$$

is a Liapounoff function because

$$\frac{d}{dt}\left(n + C + \frac{r}{2al}T^2\right) = \frac{k_0 n}{l} - \frac{grT^2}{al} \tag{16}$$

as it can be found from Eqs. (1), (2), and (3). Equation (16) shows that all solutions for $k_0 \leqq 0$ eventually reach the point O, i.e., the reactor shuts down.

If $k_0 > 0$, no Liapounoff function valid for the whole space has been found yet. In order to prove that no trajectory goes to infinity, it is sufficient to

prove that any trajectory crosses inward in a closed surface surrounding the stable equilibrium point X This surface is defined by 13 adjacent surfaces (Fig. 1).

DG*V*GPU*D	(Plane P_1 parallel to $C = 0$)
OEFF*DU*UZO	(Plane P_2: $n = 0$)
F*G*V*GVFF*	(Plane P_3: $\beta n/l = \lambda C$)
FGLF	(Ruled paraboloid RP_1)
GKLG	(Plane P_4 parallel to $T = 0$)
PBSKGP	(Plane P_5 to be defined)
SKLWFERJS	(Plane P_6: $(k_0 - \beta)(n/l) + \lambda C = 0$)
ERQE	(Plane P_7 parallel to $T = 0$)
RSTQR	(Plane P_8 parallel to $n = 0$ going through A)
UYZU	(Ruled paraboloid RP_2)
OEQTYZO	(Plane P_9:$C = 0$)
BPU*UYTSB	(Plane P_{10}:$T = 0$)
DF*G*	(Plane P_{11} parallel to $T = 0$)

All trajectories cross

P_2 because $dn/dt > 0$ on P_2
P_3 and P_8 because $dn/dt < 0$ on P_3 and P_8
P_6 and P_9 because $dC/dt > 0$ on P_6 and P_9
P_1 because $dC/dt < 0$ on P_1
P_4 and P_{10} because $dT/dt > 0$ on P_4 and P_{10}
P_7 and P_{11} because $dT/dt < 0$ on P_7 and P_{11}
RP_1 and RP_2 because

$$(d/dt)\{n + C + (r/2al)[T - (k_0/r)]^2\} = -(gr/al)[T - (k_0/r)]T \quad (17)$$

The plane P_{11} is chosen such that both points F and V* are below it. The plane P_5 is chosen such that all trajectories cross P_5 from the outside to the inside. The plane P_5 exists because $dT/dt > 0$ everywhere on PBSKGP. By moving the point A, it is possible to include on the closed surface all possible points of the phase space. Since no trajectory leaves the closed surface, all trajectories are bounded. For the case $k_0 \geq \beta$ the line OS is on the other side of OT. Then the closed surface is defined by the 10 adjacent surfaces: DG*V*GPU*D, OEFF*DU*UZO, F*G*-V*GVFF*, FGLMF, GKK*MLG, PBSTK*KGP, UYZU, OEFMK*J*TYZO, BPU*UYTSB, DF*G*. It can also be proved that all trajectories cross this closed surface inward. The shape of the closed surface does not preclude the existence of periodic solutions (limit cycle). From Eqs. (1), (2), (3), and (17) it is found that

$$\frac{d}{dt}\left[n - n_0 - n_0 \log \frac{n}{n_0} + C - C_0 + \frac{r}{2al}\left(T - \frac{k_0}{r}\right)^2\right] = -\frac{rg}{al}\left(T - \frac{k_0}{r}\right)^2 + \frac{n_0}{n}\frac{dC}{dt} \quad (18)$$

From Eq. (2)

$$C = \frac{\beta}{l\lambda}\int_0^\infty n(t - u)D(u)\,du \quad (19)$$

with $D(u) = \lambda e^{-\lambda u}$ and $\int_0^\infty D(u)\,du = 1$. Therefore, Eq. (18) becomes

$$\frac{d}{dt}\left[n - n_0 - n_0 \log \frac{n}{n_0} + C - C_0 + \frac{r}{2al}\left(T - \frac{k_0}{r}\right)^2\right] = -\frac{rg}{al}\left(T - \frac{k_0}{r}\right)^2 + \frac{n_0\beta}{l}\int_0^\infty D(u)\left[1 - \frac{n(t-u)}{n(t)}\right]du \quad (20)$$

If a periodic solution exists, Eq. (20) should be satisfied identically. The average over one period of the left-hand side is zero and the average of the right-hand side is the sum of two terms: the first one is negative and the second one is also negative since W. K. Ergen (*5*) has proved that the average of $1 - [n(t - u)/n(t)]$ is negative. Because Eq. (20) cannot be satisfied in the average by a periodic solution, the system has no periodic solutions. From Eq. (20) it is seen that the effects of delayed neutrons and Newton's law of cooling add independently in order to improve the stability of a nuclear reactor. This conclusion has been checked in many other examples (*20*).

From this analysis of the properties of the trajectories in the phase space, it has been proved that all trajectories eventually reach the only stable equilibrium point (O if $k_0 \leqq 0$ and X if $k_0 > 0$) and that periodic solutions do not exist. Therefore the stable equilibrium point is asymptotically stable in the large in the sense of Liapounoff. From this, it can be shown also that this nuclear reactor model is stable against constantly acting perturbances (total stability) and that a Liapounoff function valid everywhere for $k_0 \leqq$ as well as for $k_0 > 0$ exists (*8*).

Particular Solutions

In the discussion of stability of the solutions, algebraic surfaces defined by

$$F \equiv a'n + C + bT + (cT^2/2) = \text{const} \quad (21)$$

seem to have special properties with respect to the

solutions of the differential equations. If, for certain values of a', b, and c, the equation

$$dF/dt = mF \tag{22}$$

holds, the trajectories in the four-dimensional space n, C, T, t are on a surface

$$F = F(n_0', C_0', T_0')e^{mt} \tag{23}$$

if n_0', C_0', T_0' are the coordinates of the initial point.

Two surfaces F have been found. The first one is defined by

$$F_1 \equiv C - (\beta/al)T \tag{24}$$

with $m = -\lambda$ provided

$$\lambda = g \tag{25}$$

The second one is defined by

$$F_2 = n + [\lambda C/(\lambda - 2g)] + [rT^2/(2al)] \tag{26}$$

with $m = -2g$ provided

$$(2gl + k_0)(\lambda - 2g) + 2\beta g = 0 \tag{27}$$

The surfaces $F_1 = 0$ and $F_2 = 0$ contain the equilibrium points O and X. Any trajectory starting on $F_1 = 0$ or $F_2 = 0$ remains on the surface and any trajectory starting outside $F_1 = 0$ or $F_2 = 0$ comes closer and closer to these surfaces without crossing them (separatrix surfaces).

If both conditions (25) and (27) hold simultaneously, then

$$\frac{2\beta - k_0}{l} = 2\lambda \tag{28}$$

Equation (28) corresponds to the case of a reactor nearly twice prompt critical. In this case, n and C can be eliminated between Eq. (3) and

$$F_1 = F_1(n_0', C_0', T_0')e^{-gt} \tag{29}$$

$$F_2 = F_2(n_0', C_0', T_0')e^{-2gt} \tag{30}$$

It is found that the temperature is solution of a Riccati equation

$$\frac{dT}{dt} = -\left(g - \frac{\beta}{l}\right)T - \frac{r}{l}\frac{T^2}{2} + a\left(n_0' - C_0' + \frac{r}{al}\frac{T_0'^2}{2}\right)e^{-2gt} + a\left(C_0' - \frac{\beta}{al}T_0'\right)e^{-gt} \tag{31}$$

which can be integrated in all cases by means of the confluent hypergeometric functions (*20*, *21*). If the initial point is on the line (called separatrix line)

$$C - \frac{\beta}{al}T = 0 \tag{32}$$

$$n - C + \frac{r}{al}\frac{T^2}{2} = 0 \tag{33}$$

Eq. (31) can be integrated immediately because the variables are separable. It is found that

$$T = \frac{k_0 T_0'}{rT_0' + (k_0 - rT_0')e^{-k_0 rt/2l}} \tag{34}$$

if $T = T_0'$ at $t = 0$.

From Eqs. (34) and (32) it is seen that the temperature and delayed neutron concentration vary monotonically to their asymptotic value. The neutron density reaches a maximum only if

$$T_0' < \beta/r, \qquad \beta > 2l\lambda$$

$$T_0' > \beta/r, \qquad \beta < 2l\lambda$$

The values of n and C can be found from Eqs. (32), (33), and (34). In Fig. 2 a three-dimensional representation of the solution is given.

The present method of derivation of particular solutions is extremely simple and does not require tedious computations. It can be generalized to more groups of delayed neutrons. The condition equations (25) and (28) have been found previously by R. S. Margulies (*4*) by means of very lengthy derivations. The results given here are more general than the ones of Margulies, who did not indicate that condition equation (27) is necessary and sufficient for the existence of F_2.

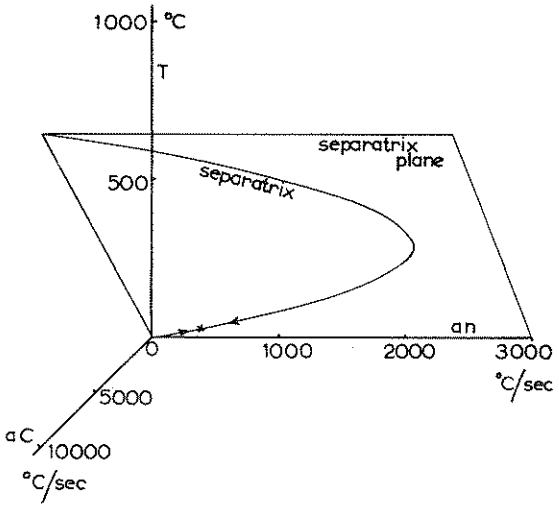

FIG. 2. Newton's law of cooling model. Separatrix. $\beta/l = 10$ sec^{-1}; $r/l = 0.02$ sec^{-1}°C^{-1}.

STEP RESPONSE—ADIABATIC MODEL: $n_0^* = g = 0$

Stability in the Large

The adiabatic model is an approximation of the Newton's law of cooling model, valid when the reactivity variation is large and the heat losses are small. It does not describe adequately the ultimate transient behavior of the reactor.

The equilibrium points are on the line

$$n = C = 0 \tag{35}$$

The properties of stability near the equilibrium points cannot be derived from the linearized equations. However, the equation

$$\frac{d}{dt}\left[n + C + \frac{(k_0 - rT)^2}{2ral}\right] = 0 \tag{36}$$

gives the equation of surfaces F_3 containing all trajectories.

$$F_3 \equiv n + C + \frac{(k_0 - rT)^2}{2ral} = B \tag{37}$$

The surface F_3 limited by the planes $n = 0$ and $C = 0$ is closed. Therefore all solutions are bounded. The temperature is an increasing function having its maximum at

$$T = \frac{k_0}{r} + \sqrt{\frac{2alB}{r}} \qquad k_0 > 0 \tag{38}$$

All the points on $n = C = 0$ above k_0/r are points of convergence and below k_0/r are points of divergence.

The elimination of C between Eqs. (1) and (37) yields

$$\begin{cases} \dfrac{dn}{dt} = \dfrac{k - \beta}{l} n + \lambda\left[B - n - \dfrac{(k_0 - rT)^2}{2ral}\right] & (39) \\ \dfrac{dT}{dt} = an & (40) \end{cases}$$

Eliminating t between Eqs. (39) and (40), it is found that the trajectories in the n,T planes are solutions of

$$l\frac{dn}{dT} = \frac{1}{an}\{(k - \beta - \lambda l)n - \lambda[(k_0 - rT)^2/2ra] + \lambda Bl\} \tag{41}$$

The singular points of Eq. (41) are an unstable saddle point at $n = 0$, $T = k_0/r - \sqrt{2alB/r}$, and a stable node at $n = 0$, $T = k_0/r + \sqrt{2alB/r}$ if $k_0 > 0$.

The curve £

$$£ \equiv \left(\frac{k - \beta}{l} - \lambda\right)n - \lambda\frac{(k_0 - rT)^2}{2ral} + \lambda B = 0 \tag{42}$$

limited by the line $n = 0$ encircles the region of the phase plane where $dn/dT > 0$. The trajectory corresponding to a sudden increase of reactivity starts at the inside of £ and reaches its maximum on £, then tends asymptotically to £ and intersects the line $n = 0$ at $T = k_0/r + \sqrt{2alB/r}$. The maximum value of n on the curve £ occurs for

$$T_m = \frac{k_0}{r} + \frac{-\beta + \sqrt{\beta^2 - 2alBr}}{r} \tag{43}$$

and is equal to

$$n_{\max} = -\frac{\lambda}{ar}(\beta - \sqrt{\beta^2 - 2alBr}) \tag{44}$$

$$\simeq \frac{\lambda lB}{\beta} \text{ if } \beta^2 \gg 2alBr \tag{45}$$

The curve £ has a maximum only if $\beta^2 \geqq 2\,alBr$ or $\beta^2 \geqq k_0^2$ if $n_0' = C_0' = T_0' = 0$, i.e., for delayed critical reactors. Since the maximum of the neutron density occurs on £, it is not larger than the maximum value of n on £. Thus Eq. (44) gives an upper bound of the neutron density for delayed critical reactors. Because the curve £ has a single maximum, the temperature at maximum neutron density is larger than T_m.

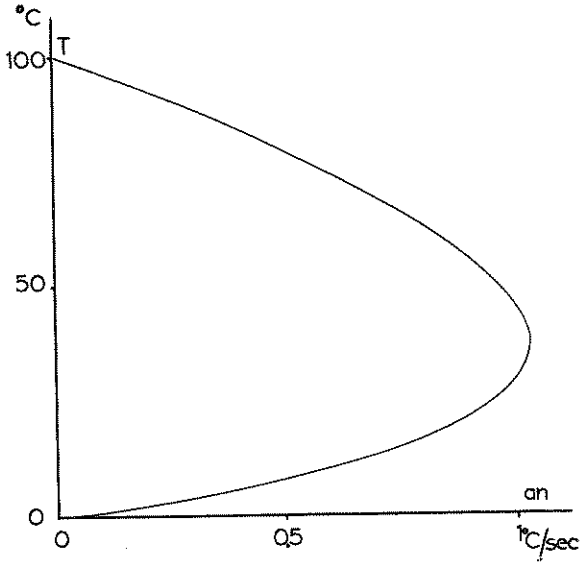

Fig. 3. Adiabatic model. Phase plane. Trajectory for $k_0/\beta = 0.5$, $\beta/l = 10$ sec^{-1}, $r/l = 0.1$ sec^{-1} °C^{-1}.

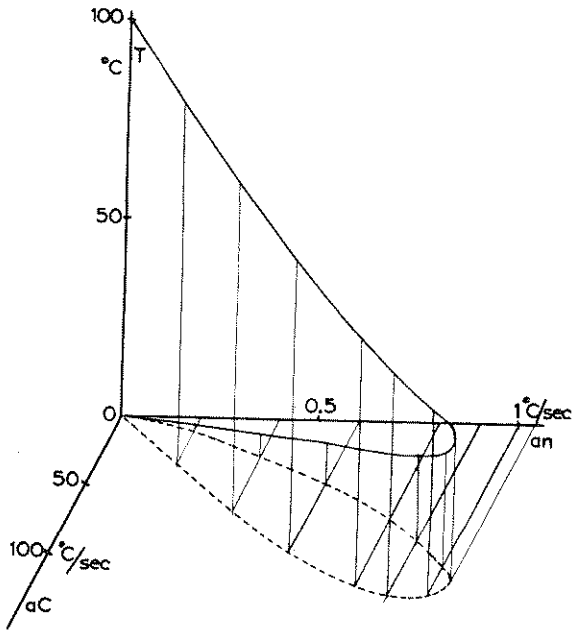

FIG. 4. Adiabatic model. Phase space. Trajectory for $k_0/\beta = 0.5$, $\beta/l = 10$ sec^{-1}, $r/l = 0.1$ sec^{-1} °C^{-1}.

PROMPT JUMP APPROXIMATION: $k_0 \ll \beta$

If the prompt jump approximation

$$ldn/dT \simeq 0 \tag{46}$$

is introduced, the curve £ is the trajectory in the n, T plane. The solution computed by means of this approximation is shown in Figs. 3 and 4. The time solution can be found easily by eliminating n between Eqs. (3) and (42) and integrating. It is found that

$$\left(\frac{E + T'}{E - T'}\right)^{\beta/rD} \times \frac{E^2}{(E^2 - T'^2)} = e^{2\lambda t} \tag{47}$$

where

$$E^2 = \frac{2alB}{r} \tag{48}$$

$$T' = T + \frac{k_0}{r} \tag{49}$$

$$T' = 0 \quad \text{at} \quad t = 0 \tag{50}$$

If $\beta/rE \gg 1$ it is found that

$$T = \frac{k_0}{r} + E \tanh \frac{\lambda r E t}{\beta} \tag{51}$$

and

$$\frac{n}{n_{\max}} = \frac{1}{\cosh^2 (\lambda r E t/\beta)} \tag{52}$$

When the maximum values of n in a reactor model with no delayed neutrons are compared to this model, it is found that the amplitude of the burst in the latter is reduced by at least a factor of one hundred and the time constant is increased by a factor of fifty.

The prompt jump approximation valid for delayed critical reactors with short neutron lifetime can also be used in the particular cases previously discussed (Eq. 21 and following) in order (*20*) to derive more properties of the solution.

FAST ACCIDENTS: $k_0 > \beta$

When $k_0 > \beta$, the curve £ does not have a maximum. However, in this case, the maximum value of n can be found because the delayed neutrons are overrun by the prompt neutrons. Then Eq. (1) becomes approximately

$$dn/dt = [(k - \beta)/l]n \tag{53}$$

and Eq. (41) becomes

$$dn/dT = (k - \beta)/la \tag{54}$$

the solution (*9*) of which is (Fig. 5)

$$n + \frac{r}{2al}\left(T - \frac{k_0 - \beta}{r}\right)^2 = \frac{(k_0 - \beta)^2}{2alr} \tag{55}$$

If $n = T \simeq 0$ at $t = 0$, it is found that

$$n_{\max} = \frac{(k_0 - \beta)^2}{2lar} \quad \text{for } T = \frac{k_0 - \beta}{r} \tag{56}$$

Equation (55) is valid only during the time for which n is very large. When time elapses, the delayed

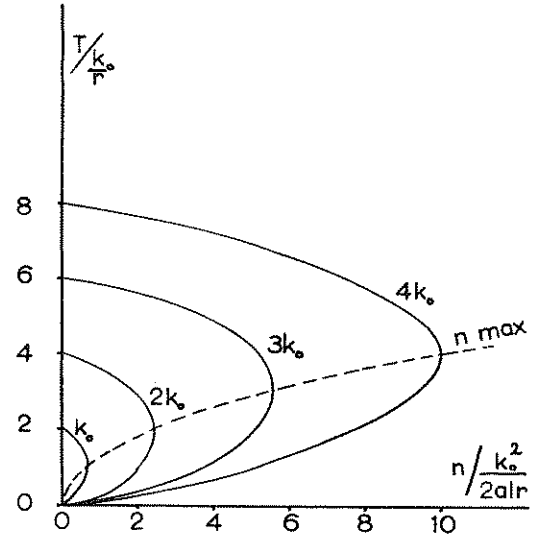

FIG. 5. Adiabatic model with no delayed neutrons. Phase plane. Trajectories in the case of large step reactivity variations and zero power initial conditions.

neutrons generated during the first part of the transient and the heat losses must be taken into account.

Application—The "Loss-of-Coolant" Accident

If a reactor in steady state is operating at the power n_0' and experiences a "loss-of-coolant" accident, the increase of temperature can be computed by means of Eq. (37). If $n = n_0'$, $C = (\beta n_0'/l\lambda)$, $T = T_0' = (k_0/r)$ are the initial conditions, the increase of temperature is

$$\Delta T = \sqrt{\frac{2al}{r}(n_0' + C_0')} \simeq \sqrt{\frac{2a\beta n_0'}{r\lambda}} \qquad (57)$$

and is independent of the neutron lifetime. The neutron density is decreasing continuously and the reactor shuts down. If, at the time of the accident, the reactor is scrammed (reactivity decrease: δk), the temperature increase is

$$\Delta T = -\frac{\delta k}{r} + \sqrt{\left(\frac{\delta k}{r}\right)^2 + \frac{2\,al}{r}(n_0' + C_0')} \qquad (58)$$

If $\delta k/r$ is large, the increase in temperature is

$$\Delta T \simeq \frac{al(n_0' + C_0')}{\delta k} \simeq \frac{a\beta n_0'}{\delta k \lambda} \qquad (59)$$

Step Response—The Constant Power Removal Model: $n_0^* \neq 0$, $g = 0$

The constant power removal model is not an approximation of the Newton's law of cooling model but a topologically different model. It is based on the usually wrong assumption that power is removed at a constant rate independently of the temperature. For sake of brevity, only the principal results (*20*) will be stated here. The phase space n, C, T contains only one equilibrium point (always stable) corresponding to a nonzero power independent of the reactivity. The tendency toward oscillation near the equilibrium point is enhanced by a large operating power and a large temperature coefficient of reactivity.

All trajectories are bounded and no periodic solution exists. The proof of boundedness stems from the consideration of Liapounoff's functions valid in subspaces of the phase space. Therefore the equilibrium point corresponding to a finite nonzero power is asymptotically stable in the large in the sense of Liapounoff. This conclusion is clearly opposite to the well-known fact that a reactor can be shut down by scram action.

Ramp Response—Adiabatic and Constant Power Removal Models: $g = 0$

If the variation of reactivity is linear,

$$k = k_0 - rT + yt \qquad (60)$$

the system of differential equations is nonautonomous and topological methods are difficult to apply. However, in the case of adiabatic and constant power removal models, it is possible to make a transformation of variables and obtain a system of differential equations with constant coefficients If the dependent variable T is replaced by $-k/r$ Eqs. (1), (2), and (3) become

$$\frac{dn}{dt} = \frac{-r(-k/r) - \beta}{l} n + \lambda C \qquad (61)$$

$$dC/dt = (\beta n/l) - \lambda C \qquad (62)$$

$$d(-k/r)/dt = a(n - n_0^{**}) \qquad (63)$$

where

$$n_0^{**} = n_0^* + (y/ra) \qquad (64)$$

This system of differential equations is identical to the one corresponding to the step response of the constant power removal model studied above. It can be shown (*20*) that n, C, and k are bounded and tend asymptotically to a constant value and that no periodic solution exists. The oscillatory character of the solution increases with n_0^* and y. If there were no delayed neutrons, the solution would be periodic.

Conclusion

The topological methods have been applied to the problems of dynamic behavior of homogeneous nuclear reactors with negative temperature coefficient. It is shown that all solutions are bounded and reach asymptotically an equilibrium value for step and ramp variation of reactivity in the three models of reactor heat transfer: Newton's law of cooling, adiabatic, and constant power removal models. The problem of ramp variation of reactivity in the adiabatic and constant power removal models is shown to be equivalent to the problem of step variation of reactivity in the constant power removal model.

Note added in proof: The second method of Liapounoff has been recently extended by the first author ["Un critère de stabilité non-linéaire des réacteurs nucléaires." *Bull. Acad. Roy. Belg. Cl. Sci.* **45**, 102–107 (1959)]. It is shown by purely analytical means that a reactor which is stable in the small for all values of the operating power is also absolutely stable in the large. The reactor models dealt with in the present paper are very simple examples of reactors stable for all power n_0.

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