Reprinted from NUCLEAR SCIENCE AND ENGINEERING, Volume 7, No. 6, June 1960
Academic Press Inc. *Printed in U.S.A.*

NUCLEAR SCIENCE AND ENGINEERING: 7, 533–540 (1960)

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Boundedness and Stability in Nonlinear Reactor Dynamics

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Received April 11, 1960

A novel formulation of the problem of boundedness and stability of the power level of a nuclear reactor describable by a nonlinear model is presented. A sufficient criterion for boundedness and stability is derived and proved to be equivalent to the criterion suggested by Welton. The criterion is illustrated by means of two examples.

NOMENCLATURE

A_1	Element of matrix M_1
$B = \pi/h$	Buckling
B_i	Element of matrix M_1
c	Specific heat
$C_{i0}(\bar{r})$, $C_i(\bar{r}, t)$	Delayed neutron precursor concentrations
$_i(E)$	Energy spectrum of delayed neutrons
F	Coolant flow
$g(t)$	Weighting function
$G(\omega)$	cosine transform of $g(t)$
$G_s(\omega)$	sine transform of $g(t)$
h	Height of reactor
$J_i = (1/4\pi)f_i(E) \cdot \int_0^\infty \int_\Omega dE' d\Omega' \nu' \Sigma_f v' \cdots$	Production operator
$K = \bar{\Omega} v \cdot \text{grad} + v\Sigma - \int_0^\infty \int_\Omega dE' d\Omega' (\Gamma_s + \Gamma_i) \cdots$	Depletion operator
l	Neutron mean lifetime
m	Mass of reactor
$M(Z, \bar{r}, t)$, $M_0(\bar{r})$, $M_i(\bar{r}, t)$, $M_s(\bar{r})$	Matrix operators
$N_0(\bar{r}, v, \bar{\Omega})$, $N(\bar{r}, t, v, \bar{\Omega})$	Neutron density
$P_1(t)$, P_0	Reactor power
Q	Neutron source density
Q_r	Heat per fission
R	Constant
u	Speed of fuel
U, dU	Volume in phase space
v	Neutron speed
V_1	Radiolytic gas volume increment
$Z(\bar{r}, t, v, \bar{\Omega})$, $Z_0(\bar{r}, v, \bar{\Omega})$	Vector variables
$Z = \left[\begin{array}{c} N(r, t, v, \Omega) \\ \hline f_i(E) C_i(\bar{r}, t) \end{array}\right]$	Column matrix
α_θ, α_v	Reactivity coefficients
β, β_i	Delayed neutron fraction
Γ_s, Γ_i	Probability of elastic or inelastic collisions
$\delta(x)$	Dirac δ function
$\theta(z, t)$, $\theta_1(t)$	Temperature
λ_i	Decay constant
$1/\mu$	Time delay for bubble formation
ν	Neutrons per fission
$1/\sigma$	Radiolytic gas residence time
Σ, Σ_f	Cross sections
$\phi(z, t)$, $\phi_0(z)$	Neutron flux

1. INTRODUCTION

The dynamics of power nuclear reactors can be described by nonlinear integrodifferential equations. The problem of boundedness and stability of the solutions of these equations has been examined quite generally by Welton (*1*).

The present paper is a novel formulation of the same problem with emphasis on mathematical rigor. On the basis of a reactor model, similar to the one assumed by Welton, a general sufficient criterion of

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boundedness and stability is rigorously derived and shown to be equivalent to the criterion presented by Welton.

The criterion is first derived in terms of generalized abstract variables which simplify the mathematical complexity of the problem. It is then expressed in terms of reactor parameters, without any loss of generality, and illustrated by means of two examples.

2. NUCLEAR REACTOR MODEL

The nuclear reactor model to be considered is based on the following assumptions:

(a) The spatial distribution of fissionable or fertile materials, moderator and coolant is independent of the variables of the system.

(b) The nuclear reactor is a closed system and no time dependent signals are applied to it externally during the time interval during which the question of boundedness and stability is investigated. In other words the reactor is autonomous after $t = 0$.

Thus, the most general equations of the neutron kinetics are

$$f_i \frac{\partial N}{\partial t} = (1 - \beta)J_0 N - KN + \frac{1}{4\pi}\sum_i^m \lambda_i f_i C_i + Q \quad (2.1)$$
$$\frac{\partial C_i}{\partial t} = 4\pi\beta_i J_i N - \lambda_i f_i C_i \quad (i = 1, 2 \cdots m)$$

The functional dependence of all variables is shown in the nomenclature.

For the purposes of this discussion the source term Q will be assumed equal to zero.

3. MATRIX FORM OF NEUTRON KINETICS

The system of Eqs. (2.1) may be represented by the matrix equation

$$\partial Z/\partial t = MZ \quad (3.1)$$

where M is a $(m + 1) \times (m + 1)$ matrix operator [Eq. (3.1a) below] and Z is a column matrix with $(m + 1)$ elements [Eq. (3.1b)].

$$Z = \begin{bmatrix} N \\ f_1 C_1 \\ \vdots \\ f_m C_m \end{bmatrix} \quad (3.1b)$$

The introduction of the generalized vector variable Z allows, as it will become evident shortly, a more systematic discussion of the reactor dynamics, even though admittedly in terms of quantities which are not physically tractable.

Equation (3.1) will now be transformed into a form suitable for the purposes of this paper. From the definitions of the various elements of the matrix operator M (see nomenclature) it can be seen that the latter is a linear operator if it is assumed that Σ, Σ_f, Γ_s, Γ_i, f_i, etc., have definite values. However, the effective cross sections and the energy spectra do depend on the reactor variables N, C_i, or more generally on Z, through the effects of temperature, pressure, fission fragments concentrations, etc. Therefore, during dynamic operation of the reactor, M is a nonlinear operator which is implicitly dependent on time. Moreover, if the reactor system is not autonomous, then M depends explicitly on time also.

In view of these comments, suppose that an equilibrium solution Z_0 of the reactor equations is known. Then the matrix M resumes its static value M_s and is linear. The solution Z_0 is given by the equation

$$M_s Z_0 = 0 \quad (3.2)$$

and since M_s is a linear operator, the adjoint equation

$$M_s^* Z_0^* = 0 \quad (3.3)$$

can also be solved to yield the adjoint vector Z_0^*.

Given the adjoint vector Z_0^*, Eq. (3.1) can be transformed into

$$(\partial/\partial t) \int_U Z_0^* Z dU = \int_U Z_0^* MZ \, dU \quad (3.4)$$

where the integration extends over all the variables of phase space.

The right-hand side of Eq. (3.4) is stationary with

$$M = \begin{bmatrix} (1 - \beta)J_0 - K & \frac{1}{4\pi}\lambda_1 & - & - & - & \frac{1}{4\pi}\lambda_m \\ 4\pi\beta_1 J_1 & -\lambda_1 & 0 & - & - & 0 \\ 4\pi\beta_2 J_2 & 0 & -\lambda_2 & - & - & 0 \\ \hdashline 4\pi\beta_m J_m & 0 & 0 & - & - & -\lambda_m \end{bmatrix} \quad (3.1a)$$

respect to variations in Z if the corresponding variations of M are not included. More specifically, if the equilibrium vector Z_0 is perturbed so that $Z = Z_0 + \delta Z$ and the corresponding perturbation of the matrix operator is δM so that $M = M_s + \delta M$, then

$$\begin{aligned}\int_U Z_0^* MZ\, dU &= \int_U Z_0^*(M_s + \delta M)(Z_0 + \delta Z)\, dU \\ &= \int_U Z_0^* M Z_0\, dU + \int_U Z_0^* \delta M \delta Z\, dU \\ &\quad + \int_U Z_0^* M_s \delta Z\, dU\end{aligned} \tag{3.5}$$

The last term of Eq. (3.5) is zero provided that Z_0^* satisfies boundary conditions which are adjoint to the boundary conditions satisfied by Z_0, because

$$\int_U Z_0^* M_s \delta Z\, dU = \int_U \delta Z M_s^* Z_0^*\, dU = 0 \tag{3.6}$$

Consequently

$$(\partial/\partial t) \int_U Z_0^* Z\, dU = \int_U Z_0^* M Z_0\, dU + \epsilon_2 \tag{3.7}$$

where ϵ_2 is of second order in δZ and δM.

Equation (3.7) may be rewritten in a different form if the operator M is formally developed into a series of operators M_i:

$$M = M_0 + M_1[Z_0^*Z] + M_2[Z_0^*Z]^2 + M_3[Z_0^*Z]^3 + \cdots \tag{3.8}$$

where $[Z_0^*Z]$ is a scalar not to be confused with the functional dependencies of the operators M_i (given in the nomenclature). The operators M_i are applied to whatever is to their right, once the formal expansion (3.8) is replaced into Eq. (3.7). More specifically, operator M_i operates on $[Z_0^*Z]^i Z_0$ etc. It should be pointed out that the static value M_s of M is not the same as M_0. In fact, $M_s = M_0 + M_1[Z_0^*Z_0] + M_2[Z_0^*Z_0]^2 + \cdots$

If it is assumed that $M_i = 0$ for $i \geqq 2$, Eq. (3.7) reduces to

$$\frac{\partial}{\partial t} \int_U Z_0^* Z\, dU = \int_U Z_0^* M_0 Z_0\, dU + \int_U Z_0^* M_1[Z_0^*Z] Z_0\, dU + \epsilon_2 \tag{3.9}$$

while the just critical reactor equation is

$$0 = \int_U Z_0^* M_0 Z_0\, dU + \int_U Z_0^* M_1[Z_0^*Z_0] Z_0\, dU \tag{3.10}$$

Equation (3.10) can be interpreted as a reactivity balance. The first term on the right-hand side represents reactivity independent of nuclear or non-nuclear variables and the second is associated with the latter or the feedback reactivity of the system.

Subtracting Eq. (3.10) from (3.9) it is found that

$$\frac{\partial}{\partial t} \int_U Z_0^* Z\, dU = \int_U Z_0^* M_1[Z_0^*Z - Z_0^*Z_0] Z_0\, dU + \epsilon_2 \tag{3.11}$$

Equation (3.11) is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t} \ln \int_U Z_0^* Z\, dU &= \left[\frac{\int_U Z_0^* M_1[Z_0^*Z] Z_0\, dU}{\left[\int_U Z_0^* Z\, dU\right]^2}\right] \int_U Z_0^* Z\, dU \\ &\quad - \left[\frac{\int_U Z_0^* M_1[Z_0^*Z_0] Z_0\, dU}{\int_U Z_0^* Z\, dU \int_U Z_0^* Z_0\, dU}\right] \int_U Z_0^* Z_0\, dU + \epsilon_2'\end{aligned} \tag{3.12}$$

If Z is replaced by Z_0, in the bracketed quantities of the right-hand side of Eq. (3.12), a second-order error is introduced, provided that M_1 is a first-order quantity, which actually is inherent in the assumption that $M_i = 0$, $i \geqq 2$. Thus, Eq. (3.12) yields

$$\frac{\partial}{\partial t} \ln \int_U Z_0^* Z\, dU = \left[\frac{\int_U Z_0^* M_1[Z_0^*Z_0] Z_0\, dU}{\left[\int_U Z_0^* Z_0\, dU\right]^2}\right] \cdot \int_U [Z_0^*Z - Z_0^*Z_0]\, dU + \epsilon_2'' \tag{3.13}$$

The implication of Eq. (3.13) is that the time-dependent operator

$$H = \left[\frac{\int_U Z_0^* M_1[Z_0^*Z_0] Z_0\, dU}{\left[\int_U Z_0^* Z_0\, dU\right]^2}\right] \tag{3.14}$$

is applied on $\int_U [Z_0^*Z - Z_0^*Z_0]\, dU$. Equation (3.13) is further simplified to read

$$(\partial/\partial t) \ln P = H(P - P_0) \tag{3.15}$$

if ϵ_2'' is neglected and

$$P = \int_U Z_0^* Z\, dU$$

$$P_0 = \int_U Z_0^* Z_0\, dU$$

It is evident that P_0 is proportional to the total power of the reactor under equilibrium conditions but P cannot be physically interpreted.

4. BOUNDEDNESS OF SOLUTIONS

The boundedness of the solutions of the reactor kinetics is now established on the basis of Eq. (3.15) which has been shown to be good to the second order.

For all practical purposes, H may be assumed an integral operator with respect to time. More specifically, as shown by Welton (*1*):

$$H = \int_{-\infty}^{t} d\tau g(t - \tau) \cdots \tag{4.1}$$

where $g(t)$ is a weighting function such that $g(t) = 0$ for $t < 0$.

If $g(t)$ is Fourier transformable, then its cosine transform is

$$G(\omega) = \frac{1}{2\pi} \int_0^\infty g(\tau) \cos \omega\tau \, d\tau \tag{4.2}$$

or

$$g(t) = \int_0^\infty G(\omega) \cos \omega t \, d\omega \tag{4.3}$$

Consequently, Eq. (3.15) may be written as

$$\frac{\partial}{\partial t} \ln P(t) = \int_{-\infty}^{t} d\tau \int_0^\infty d\omega G(\omega) \cos \omega\,(t - \tau)[P(\tau) - P_0] \tag{4.4}$$

If it is assumed that

$$\int_{-\infty}^{0} | P(\tau) - P_0 | \, d\tau < \infty \tag{4.5}$$

the order of integration in Eq. (4.4) may be interchanged because both partial integrals converge uniformly. Thus, Eq. (4.4) may be analyzed into a system of three first-order differential equations

$$\frac{\partial}{\partial t} \ln P(t) = \int_0^\infty G(\omega)[q(\omega, t) + q^*(\omega, t)] \, d\omega \tag{4.6}$$

$$\frac{\partial q(\omega, t)}{\partial t} = i\omega q(\omega, t) + \frac{1}{2}[P(t) - P_0] \tag{4.7}$$

$$\frac{\partial q^*(\omega, t)}{\partial t} = -i\omega q^*(\omega, t) + \frac{1}{2}[P(t) - P_0] \tag{4.8}$$

where

$$\begin{aligned} q(\omega, t) &= \frac{1}{2} \int_0^\infty e^{i\omega t}[P(t - \tau) - P_0] \, d\tau \\ &= \frac{1}{2} \int_{-\infty}^{t} e^{i\omega(t-\tau)}[P(\tau) - P_0] \, d\tau \end{aligned} \tag{4.9}$$

$$\begin{aligned} q^*(\omega, t) &= \frac{1}{2} \int_0^\infty e^{-i\omega\tau}[P(t - \tau) - P_0] \, d\tau \\ &= \frac{1}{2} \int_{-\infty}^{t} e^{-i\omega(t-\tau)}[P(\tau) - P_0] \, d\tau \end{aligned} \tag{4.10}$$

A first integral of the system of Eqs. (4.6) to (4.8) is

$$\begin{aligned} L = P(t) - P_0 - P_0 \ln \frac{P(t)}{P_0} \\ - 2 \int_0^\infty G(\omega) \, | q(\omega, t) |^2 \, d\omega = \text{constant} \end{aligned} \tag{4.11}$$

as can be easily verified by evaluating the time derivative of L.

The function $P(t) - P_0 - P_0 \ln[P(t)/P_0]$ is always positive and has a minimum equal to zero at $P(t) = P_0$. Consequently, the only meaningful value of the time invariant L, which plays the role of kinetic energy of the system, is

$$0 < L < \infty \tag{4.12}$$

A sufficient condition for (4.12) to be true is

$$\int_0^\infty G(\omega) \, | q(\omega, t) |^2 \, d\omega \leqq 0 \tag{4.13}$$

This inequality must be true regardless of the value of $q(\omega, t)$, which implies that

$$G(\omega) \leqq 0 \tag{4.14}$$

The actual value of L can be found by considering the instant of time $t = 0$ when

$$\begin{aligned} L = L_0 = P(0) - P_0 - P_0 \ln \frac{P(0)}{P_0} \\ - 2 \int_0^\infty G(\omega) \, | q(\omega, 0) |^2 \, d\omega \end{aligned} \tag{4.15}$$

Evidently, if (4.13) and (4.14) are true, the integral $\int_0^\infty | q(\omega, 0) |^2 d\omega$ is finite. This integral can be transformed by means of Parseval's theorem to read

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$$\begin{aligned} \int_0^\infty | q(\omega, 0) |^2 \, d\omega \\ = \frac{1}{4} \int_0^\infty d\omega \left| \int_{-\infty}^{0} e^{i\omega\tau}[P(\tau) - P_0] \, d\tau \right|^2 \\ = \frac{1}{8} \int_{-\infty}^{0} [P(\tau) - P_0]^2 \, d\tau < \infty \end{aligned} \tag{4.16}$$

Consequently, if $G(\omega) \leqq 0$ and $P(t) - P_0$ is square integrable and absolutely convergent in the time interval $\{-\infty, 0\}$, the function L has a finite positive value which is conserved for all times $0 < t < \infty$ because the system is autonomous. Therefore, $P(t)$ remains bounded.

Actually, conditions (4.5) and (4.16) are less restrictive than they appear to be because the time $t = 0$ is arbitrary and can be taken as far in the past history of the reactor as it is necessary for these conditions to be true. In particular, condition (4.13) is more of a mathematical rather than physical character on account of the limited memory that the reactor possesses. However, this mathematical restriction is very important with respect to the question concerning the stability of the solutions, as it will be shown in the next section.

Finally, the restrictions on the initial values of $P(t)$ for $t < 0$ are sufficient and may not be necessary. However, the boundedness problem has no meaning outside the context of initial values and any statement on stability without reference to initial values is likely to be somewhat incomplete.

5. STABILITY OF SOLUTIONS

The stability of the solutions of Eq. (3.15) may be established as follows:

The first integral L of the system of Eqs. (4.6) to (4.8) is valid for all values of time $t > 0$. Consequently for $t = \infty$ the boundedness requirement becomes

$$\int_0^\infty | q(\omega, \infty) |^2 \, d\omega < \infty \tag{5.1}$$

or

$$\int_{-\infty}^\infty | q(\omega, \infty) |^2 \, d\omega < \infty \tag{5.2}$$

because of symmetry. Since $P(\tau) - P_0$ is square integrable, condition (5.2) may be written as

$$\int_{-\infty}^\infty [P(\tau) - P_0]^2 \, d\tau < \infty \tag{5.3}$$

which implies that

$$\int_0^\infty [P(\tau) - P_0]^2 \, d\tau < \infty,$$

or

$$\lim_{t\to\infty} P(t) = P_0$$

Thus, the solutions are bounded and stable if $G(\omega) \leqq 0$ and conditions (4.5) and (4.16) are satisfied.

6. FORMULATION OF STABILITY CRITERION IN TERMS OF REACTOR MEASURABLE PARAMETERS

From the definition of H [Eq. (3.14)] and $g(t)$ [Eq. (4.1)], it is evident that the boundedness and stability criterion $G(\omega) \leqq 0$ requires the evaluation of the operator M_1. However, even though, in principle, it is possible to expand the operator M into a power series of $Z_0{}^*Z = N_0{}^*N + \sum_i^m f_i^2 C_{i0}^* C_i$ [Eq. (3.8)] and thus find M_1 ; this is not practical. Strictly speaking, the elements of M depend directly on $N_0{}^*N$ rather than $Z_0{}^*Z$ and, therefore, Eq. (3.8) would be very involved if the indicated procedure were to be carried out. Without any loss of generality, the results of the previous discussion can be reduced to a more practical form through the following procedure.

Consider the linear transformation

$$Z' = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \alpha I \end{array}\right] \left[\begin{array}{c} N \\ \hline f_i\, C_i \end{array}\right] = TZ \tag{6.1}$$

where α is a parameter which can take any arbitrary numerical value and I is the $m \times m$ unit matrix. In terms of the new vector variable Z', the reactor dynamics are given by the equation

$$(\partial Z'/\partial t) = TMT^{-1}Z' = M'Z' \tag{6.2}$$

The transformed operator M' can presumably be formally expanded into a power series

$$M' = M_0' + M_1'[Z_0'^* Z'] \tag{6.3}$$

where $M_0' = TM_0T^{-1}$, $M_1' = TM_1T^{-1}$.

Since the only elements of matrix M [Eq. (3.1a)] which depend on the reactor variables N, C_i are the elements of its first column, the matrix M_1 will necessarily be of the form

$$M_1 = \begin{bmatrix} A_1 & 0 & - & - & 0 \\ B_1 & 0 & - & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ B_m & 0 & - & - & 0 \end{bmatrix} \tag{6.4}$$

and furthermore

$$M_1' = \begin{bmatrix} A_1 & 0 & - & - & 0 \\ \alpha B_1 & 0 & - & - & 0 \\ - & - & - & - & - \\ - & - & - & - & - \\ \alpha B_m & 0 & - & - & 0 \end{bmatrix} \tag{6.5}$$

The criterion of stability is applicable for any variable Z' because the introduction of the transformation (6.1) amounts only to a scale change of the reference axis of Z. Therefore, it is applicable even when $\alpha \to 0$. But when $\alpha \to 0$, $Z_0{}^* \to N_0{}^*$, $Z' \to N$, $M_1' \to A_1$, and consequently

$$H \rightarrow \frac{\int_U N_0^* A_1 [N_0^* N_0] N_0 \, dU}{\left[\int_U N_0^* N_0 \, dU \right]^2} \tag{6.6}$$

The weighting function $g(t)$ is now derived from Eq. (6.6).

The evaluation of the criterion of boundedness and stability in terms of Eq. (6.6) is a tremendous improvement over the original H because the vector Z_0 has been replaced by the neutron density N_0 and the matrix M_1 by one element A_1. The element A_1 is the variation of the operator $(1 - \beta)J_0 - K$ with respect to the neutron density. The criterion thus becomes the same as the one that would have been found if delayed neutron precursors were omitted.

7. APPLICATIONS

The previous results will be illustrated by means of two examples.

a. Circulating Fuel Reactor

For simplicity, a one-group, bare, one-region, cylindrical reactor with a homogeneous circulating fuel is considered (Fig. 1).

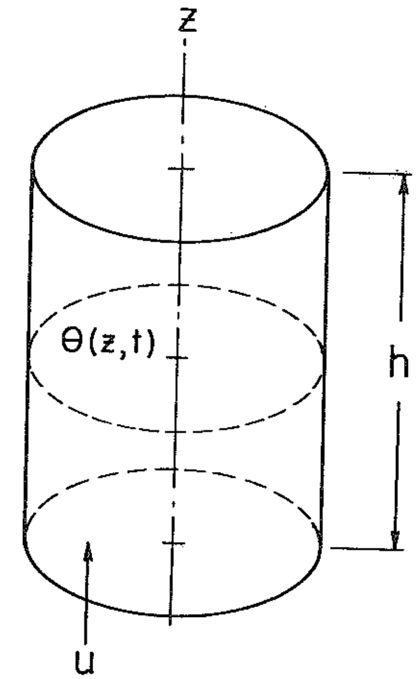

Fig. 1.
Circulating fuel reactor

If $\theta(z, t)$ is the temperature along a plane perpendicular to the fuel flow, the variation Δ of the term $(1 - \beta)J_0 - K$ of the matrix M, to first order, is

$$\Delta = (1 - \beta) \frac{f_0(E)}{4\pi} \int_0^\infty dE' \int_\Omega d\Omega' v' v' \frac{\partial \Sigma_f}{\partial \theta} \theta - v \frac{\partial \Sigma}{\partial \theta} \theta + \int_0^\infty dE' \int_\Omega d\Omega' \frac{\partial(\Gamma_s + \Gamma_i)}{\partial \theta} \theta \tag{7.1}$$

This expression, for one-group isotropic flux, reduces to

$$\Delta = \alpha_\theta \theta(z, t) \tag{7.2}$$

where α_θ = constant.

The dependence of the temperature on the flux may be described by the equation

$$c \frac{\partial \theta(z, t)}{\partial t} = Q_r \Sigma_f \phi(z, t) \tag{7.3}$$

or

$$\theta(z, t) = \frac{Q_r \Sigma_f}{c} \int_0^{z/u} \phi(z - u\tau, t - \tau) \, d\tau = \frac{Q_r \Sigma_f}{c} \int_0^\infty \phi(z - u\tau, t - \tau) \, d\tau \tag{7.4}$$

since $\phi(z, t) = 0$ for $z < 0$. Consequently

$$\alpha_\theta \theta(z, t) = \frac{Q_r \Sigma_f}{c} \alpha_\theta \int_0^\infty d\tau \int_0^h dz' \frac{\delta(z - u\tau - z')}{\phi_0^*(z')} \cdot \{\phi_0^*(z') \phi(z', t - \tau)\} \tag{7.5}$$

In view of the discussion of Section (6), the combination of Eqs. (6.5), (7.2), (7.5), and (6.6) yields

$$A_1 = \frac{Q_r \Sigma_f}{c} \cdot \alpha_\theta \int_0^\infty d\tau \int_0^h dz' \frac{\delta(z - u\tau - z')}{\phi_0^*(z')} \cdots \tag{7.6}$$

$$H = \frac{Q_r \Sigma_f}{c} \cdot \left\{ \alpha_\theta \int_0^\infty d\tau \frac{\int_0^h dz \phi_0^*(z) \int_0^h dz' \frac{\delta(z - u\tau - z')}{\phi_0^*(z')} [\phi_0^*(z') \phi_0(z')] \phi_0(z')}{\left[\int_0^h \phi_0^*(z) \phi_0(z) \, dz \right]^2} \right\}$$

$$= \frac{Q_r \Sigma_f}{c} \alpha_\theta \int_0^\infty d\tau \frac{\int_{u\tau}^h dz \phi_0^*(z) \phi_0^2(z - u\tau)}{\left[\int_0^h \phi_0^*(z) \phi_0(z) \, dz \right]^2} \tag{7.7}$$

where the flux ϕ_0 instead of the density N_0 has been used in the definition of A_1 and H. This is an allowable substitution since $\phi_0 \sim N_0$ in a one-group reactor. If $\phi_0(z) = \phi_0^*(z) = \phi_0 \cos(Bz - \pi/2)$,

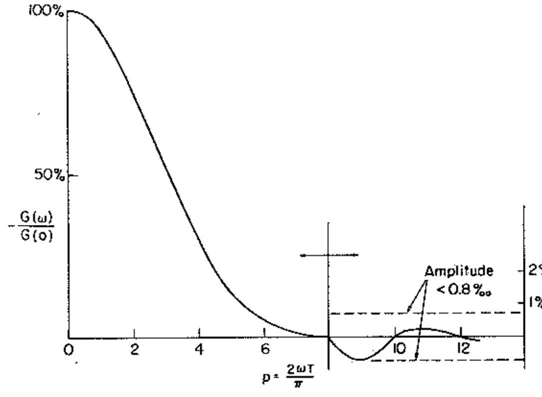

FIG. 2.
Normalized cosine transform

Eq. (7.7) becomes

$$H = A \int_0^\infty d\tau \left(1 + \cos \frac{\pi\tau}{T}\right)^2 \cdots \quad \tau < T \qquad (7.8)$$

where $A = (4Q_r\Sigma_f\alpha_\theta/3\pi c\phi_0 h)$, $T = h/u$. Evidently the weighting function $g(t)$ [Eq. (4.1)] and its cosine and sine transforms are

$$g(t) = \begin{cases} A\left(1 + \cos \frac{\pi t}{T}\right)^2 & t < T \\ 0 & t > T, t < 0 \end{cases} \qquad (7.9)$$

$$G(\omega) = \frac{3AT}{\pi} \frac{1}{(1 - p^2)(4 - p^2)} \frac{\sin \pi p}{\pi p} \qquad (7.10)$$

$$G_s(\omega) = \frac{3AT}{\pi} \frac{[1 - \cos \pi p - 2p^2 - \frac{2}{3}p^2(1 - p^2)]}{\pi p(1 - p^2)(4 - p^2)} \qquad (7.11)$$

where $p = \omega T/\pi$.

Figure 2 is a normalized plot of $G(\omega)$. Assuming the temperature coefficient $\alpha_\theta < 0$ it is observed that $G(\omega)$ is negative for $|\omega| < 3\pi/T$ but has an alternating sign for $|\omega| > 3\pi/T$ even though its magnitude in the latter interval is negligibly small. Consequently the sufficient criterion is not met, unless the values of $G(\omega)$ for $|\omega| > 3\pi/T$ are not taken into account.

The inadequacy of the sufficient criterion $G(\omega) \leqq 0$ to reveal a more definite answer about the nonlinear stability of the circulating fuel reactor is partly due to the detailed model which has been analyzed. If Ergen's model (*2*) had been considered, then

$$g(t) \sim -[1 - t/T][u(t) - u(t - T)] \qquad (7.12)$$

$$G(\omega) \sim -\frac{2\cos^2(\omega T/2)}{\omega^2 T} < 0 \qquad (7.13)$$

Equation (7.13) implies that the criterion $G(\omega) \leqq 0$ is applicable for all frequencies ω and the reactor is stable.

The linearized version of the kinetics equations leads also to a stable reactor. This fact can be shown as follows.

The criterion of stability is that the function

$$-\frac{\int \phi_0^*(z)\phi_0(z)\,dz}{i\omega}[G(\omega) - iG_s(\omega)]$$

does not encircle the point $(-1, 0)$ of the $G(s)$ plane or that the function

$$\frac{\omega}{\int \phi_0^*(z)\phi_0(z)\,dz} + G_s(\omega)$$

does not change sign for $0 < |\omega| < \infty$. Some elementary considerations easily reveal that $G_s(\omega)$ has the same sign as ω for all values of ω and is finite everywhere if $\alpha_\theta < 0$. Therefore the linearized reactor model is stable.

b. KEWB REACTOR

Consider the KEWB reactor (*3*). For simplicity assume a lumped parameter model. Under conditions of constant power removal, the dynamics of the reactor are describable by the following set of equations

$$l\frac{d}{dt}\ln\frac{P_1}{P_0} = \alpha_\theta\,\theta_1 + \alpha_v\,V_1 \qquad (7.14)$$

$$mc\frac{d\theta_1}{dt} + 2Fc\theta_1 = P_1 - P_0 \qquad (7.15)$$

$$\frac{1}{\mu}\frac{d^2V_1}{dt^2} + \left(1 + \frac{\sigma}{\mu}\right)\frac{dV_1}{dt} + \sigma V_1 = R(P_1 - P_0) \qquad (7.16)$$

provided that the delayed neutron precursors are omitted. This omission does not affect the application of the criterion derived in this paper, as is shown in Section 6.

In terms of Laplace transforms, Eqs. (7.15) and (7.16) may be written as

$$\alpha_\theta\,\bar\theta_1(s) + \alpha_v\,\bar V_1(s) = \left[\frac{A_\theta}{s + \omega_0} + \frac{A_v}{(s + \mu)(s + \sigma)}\right]\overline{[P_1 - P_0]} = G(s)\overline{[P_1 - P_0]} \qquad (7.17)$$

where $A_\theta = (\alpha_\theta/mc)$, $A_v = \alpha_v R\mu$, $\omega_0 = 2F/m$, and $\overline{[P_1 - P_0]}$ the Laplace transform of $[P_1 - P_0]$. The condition of nonlinear stability for this space independent model reduces to

$$G(j\omega) + G(-j\omega) \leqq 0$$

or

$$[A_\theta\omega_0 - A_v]x^2 + [A_\theta\omega_0(\sigma^2 + \mu^2) + A_v(\sigma\mu - \omega_0^2)]x + \sigma\mu\omega_0[A_\theta\sigma\mu + A_v\omega_0] < 0 \quad (7.18)$$

where $\omega^2 = x$. Condition (7.18) is true for all x if

$$\begin{aligned} A_\theta\omega_0 - A_v &\leqq 0 \\ A_\theta\omega_0(\sigma^2 + \mu^2) + A_v(\sigma\mu - \omega_0^2) &\leqq 0 \\ A_\theta\sigma\mu + A_v\omega_0 &\leqq 0 \end{aligned} \quad (7.19)$$

If $\sigma \sim \omega_0$ and $\mu \gg \sigma$, conditions (7.19) reduce to

$$\begin{aligned} A_\theta &< 0 \\ A_\theta\omega_0 < A_v &< -A_\theta\mu \end{aligned} \quad (7.20)$$

It is interesting to compare conditions (7.19) with the ones that would have been derived from an absolutely stable linear model. The criterion of stability is that the function

$$1 - \frac{P_0}{ls} G(s)$$

has all its zeros in the left half s plane. The conditions for this to be true can be derived from any of the well known stability criteria for linear systems and are

$$\begin{aligned} A_\theta &< 0 \\ A_\theta(\sigma + \mu) + A_v &\leqq 0 \\ A_\theta\sigma\mu + A_v\omega_0 &\leqq 0 \\ A_\theta\left[\frac{2\mu\sigma\omega_0^2}{\sigma + \mu + \omega_0} + \omega_0(\sigma^2 + \mu^2)\right. &\left. + A_v(\sigma\mu - \omega_0^2)\right] \leqq 0 \end{aligned} \quad (7.21)$$

In case $\omega_0 \sim \sigma$, $\mu \gg \sigma$, conditions (7.21) reduce to

$$\begin{aligned} A_\theta &< 0 \\ A_v &< -A_\theta\mu \end{aligned} \quad (7.22)$$

It is concluded that conditions (7.22) derived for the linear model are less restrictive than (7.20).

8. CONCLUSIONS

The sufficient criterion $G(\omega) \leqq 0$ for boundedness and stability of the power level of autonomous nuclear reactors is identical to Welton's criterion as can be shown very easily by means of a partial integration. However, the present derivation is different in many respects.

More specifically, the development of the problem is based on a system of first-order differential equations as contrasted to a second-order equation considered by Welton.

The stability of $P(t)$ is established explicitly by elementary methods, with proper reference to initial values and without use of any irreversibility arguments.

The introduction of the generalized variable Z leads to a homogeneous Eq. (3.14) which includes the delayed neutron sources. This is true to the extent that second-order effects can be neglected.

ACKNOWLEDGMENT

One of the authors (J. D.) is indebted to the Institut Interuniversitaire des Sciences Nucleaires, Belgium, for a grant which made this study possible.

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