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Effect of Delayed Neutrons on Nonlinear Reactor Stability

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The effect of delayed neutron precursors on the stability of nuclear reactors described by nonlinear equations is investigated. It is shown by means of Liapunov's second method that a sufficient condition of stability is that the reactor be stable without delayed neutron precursors. An illustrative example is included.

INTRODUCTION

The effect of delayed neutron precursors on the stability of neutron chain reactors has been investigated by several authors (*1*, *2*). In cases of autonomous reactors, with or without special nonlinearities, it has been found that delayed neutrons have a stabilizing influence on the dynamic behavior of the reactor.

The present paper is an attempt to generalize the previous results to reactors with any order nonlinearities and reactors which are nonautonomous. In the first section the dynamic equations are arranged in a convenient matrix form. In the second section the effect of the delayed neutrons on the stability of the reactor is investigated by means of Liapunov's second method (*3*). It is proved that a sufficient condition of stability is that the reactor be stable without delayed neutrons. In the third section the value of this assertion is illustrated by means of a particular reactor with two temperature coefficients. Finally, the paper is closed by a discussion of the derived results and their implications.

REACTOR MODEL

For mathematical expediency consider an unreflected reactor. Actually, in principle, there is no difficulty in analyzing a reflected reactor, but the problem becomes unnecessarily complex.

The reactor dynamics is describable by two sets of time dependent equations. The first is related to the neutron balance and the second to the nonnuclear variables. Both sets of equations are subsequently derived and reduced to a convenient matrix form.

NEUTRON KINETICS

Assume the reactor fuel is stationary. The kinetics equations can be derived from the time dependent transport equation. More specifically, the time behavior of the directional flux density and the delayed neutron concentrations are described by the following equations (*4*):

$$\frac{\partial N}{\partial t} = [(1 - \beta\ )J_0 - K]\,N + \frac{1}{4\pi}\sum_i^m \lambda_i f_i C_i \qquad (1)$$

$$f_i \frac{\partial C_i}{\partial t} = 4\pi\beta_i J_i N - \lambda_i f_i C_i \qquad i = 1, 2 \cdots m \qquad (2)$$

where

$C_i = C_i(\bar{r}, t)$	Delayed neutron precursor concentration
$f_i = f_i(E)$	Energy spectrum of neutrons of ith species
$N = N(\bar{r}, v, \bar{\Omega}, t)$	Neutron directional flux density
v	Neutron speed
β, β_i	Delayed neutron fractions

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$\Gamma_s(\bar{r}, E', E, \bar{\Omega}' \cdot \Omega, t)$	Total cross section for scattering from energy E' and direction $\bar{\Omega}'$ to E and $\bar{\Omega}$
λ_i	Decay constant
ν	Fission neutron yield
$\Sigma(\bar{r}, E, t)$	Total cross section for absorption
$\Sigma_f(\bar{r}, E, t)$	Fission cross section

$$J_i = \frac{1}{4\pi} f_i \int_0^\infty \int_\Omega dE' \, d\bar{\Omega}' \nu' \, \Sigma_f \, v'$$

Neutron production operator

$$K = \bar{\Omega} v \cdot \text{grad} + v\Sigma - \int_0^\infty \int_\Omega dE' \, d\bar{\Omega}' \Gamma_s$$

Neutron depletion operator

In the definition of the various cross sections a time dependency is introduced to account for the effects of the nonnuclear variables and external disturbances on the neutron population.

Equations (1) and (2) are very cumbersome to use. However, they can be reduced to a more convenient form by formally integrating out all the phase space dependencies, as indicated by Henry (*4*). To this effect, define the parenthesis (F, G) as

$$(F, G) = \int_U FG \, dU \tag{3}$$

where the integral is taken over the entire volume of the phase space. Furthermore, introduce the change of variable

$$N = A(t) N_0(\bar{r}, v, \bar{\Omega}) \tag{4}$$

Thus, the original equations may be rearranged to yield

$$\frac{dA(t)}{dt} = \frac{\rho - \bar{\beta}}{\Lambda} A(t) + \sum_i^m \lambda_i C_i(t) \tag{5}$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_e^{\,i}}{\Lambda} A(t) - \lambda_i C_i(t) \qquad i = 1, 2 \cdots m \tag{6}$$

where

$$\Lambda = (N_0^*, N_0)$$

$$\bar{\beta} = \beta(N_0^*, J_0 N_0)$$

$$\Lambda C_i(t) = (1/4\pi)(N_0^*, f_i C_i)$$

$$\rho = (N_0^*, [J_0 - K] N_0)$$

$$\beta_e^{\,i} = \beta_i(N_0^*, J_i N_0)$$

$$N_0^* = \text{adjoint of } N_0$$

For a complete discussion of the physical meaning of the previous transformations, the reader is referred to reference *4*. For the purposes of this presentation, it suffices to note the following:

(*a*) The change of variable, introduced in Eq. (4), is no restriction on N. It is merely a definition. However, in many practical cases, it can be shown that $N_0(\bar{r}, v, \bar{\Omega})$ may be taken as the fundamental mode of the critical reactor (*4*).

(*b*) The derived Eqs. (5) and (6) are similar in form to the conventional kinetics equations but for a fundamental difference. The coefficients are implicit or explicit functions of time, through the effects of nonnuclear variables such as temperature, pressure, voids, etc. or extraneous disturbances, respectively.

Nonnuclear Dynamics

Nonnuclear dynamics refers to the relationships between reactor power, temperatures, pressures, flows, voids, etc. throughout the reactor core. These relationships are derived from the equations of conservation of energy, momentum and mass.

For the purposes of this presentation it is not necessary to derive specific equations for the nonnuclear dynamics. Only the form of the equations is adequate.

To this effect, consider a typical nonnuclear variable denoted by $T_k(\bar{r}, t)$. This variable may be averaged over a number of regions in the reactor and approximated as closely as desired by a set of functions $T_j(t)$. The latter functions are, in general, solutions of equations of the form:

$$[dT_j(t)/dt] = F_j[T_l(t), A(t)] \qquad j, l = 1, 2 \cdots n \tag{7}$$

The set of equations (7) describes the nonnuclear dynamics. It is assumed to be of the first order. This is always possible because, if the equations were not of first order, one could perform an appropriate change of variables and reduce them to first order. Also, the functions F_j are assumed independent of the delayed neutron precursors since there is no physical process which links $T_j(t)$ and $C_i(t)$.

Matrix Form of Reactor Dynamics

Equations (5), (6), and (7) give the complete dynamic picture of the reactor. In this section, they are reduced to a compact matrix form as follows.

The coefficients $(\rho - \bar{\beta})/\Lambda$, $\beta_e^{\,i}/\Lambda$, and the functions F_j are expanded into power series of the nuclear and nonnuclear variables. Thus, a new system of dynamic equations results:

$$\frac{da(t)}{dt} = \frac{\rho_0 + \rho(t) - \beta_0}{\Lambda_0} a(t) + \sum_i^m \lambda_i c_i(t) + A_0 \sum_j^n k_j(t)\theta_j(t) + \frac{\rho(t)}{\Lambda_0} A_0 + \eta(a, \theta_l) \tag{8}$$

$$\frac{dc_i(t)}{dt} = \frac{\beta_{e0}^i}{\Lambda_0} a(t) - \lambda_i c_i(t) + A_0 \sum_j^n q_{ij}(t)\theta_j(t) + \zeta_i(a, \theta_l) \tag{9}$$

$$\frac{d\theta_j(t)}{dt} = r_j(t)a(t) + \sum_l^n \sigma_{jl}(t)\theta_l(t) + \mu_j(a, \theta_l) \tag{10}$$

where

$a(t) = A(t) - A_0$
$c_i(t) = C_i(t) - C_0$
$\theta_j(t) = T_j(t) - T_{j0}$
$\rho_0 + \rho(t), \bar{\beta}_0, \beta_{e0}^i, \Lambda_0$ = values of coefficients at $T_j = T_{j0}$ and $A = A_0$
ρ_0 is so chosen that $\rho_0 - \bar{\beta}_0 + \sum_i^m \beta_{e0}^i = 0$

$$k_j(t) = \frac{\partial}{\partial T_j}\left[\frac{\rho - \bar{\beta}}{\Lambda}\right]_{T_j = T_{j0}}$$

$$q_{ij}(t) = \frac{\partial}{\partial T_j}\left[\frac{\beta_e^i}{\Lambda}\right]_{T_j = T_{j0}}$$

$$r_j(t) = \frac{\partial}{\partial A}[F_j(T_i(t), A(t))]_{T_i = T_{i0}, A = A_0}$$

$$\sigma_{jl}(t) = \frac{\partial}{\partial T_l}[F_j(T_i(t), A(t))]_{T_i = T_{i0}, A = A_0}$$

η, ζ_i, μ_j = second-order quantities in $a(t)$ and $\theta_l(t)$.
All the time-dependent coefficients are assumed uniformly bounded.

The set of equations (8) to (10) is equivalent to the matrix equation

$$\frac{d}{dt}\begin{bmatrix} C \\ P \end{bmatrix} = \begin{bmatrix} -\Delta & B \\ E & D \end{bmatrix}\begin{bmatrix} C \\ P \end{bmatrix} + \frac{A_0}{\Lambda_0}[\rho(t)] + Q \tag{11}$$

or

$$\frac{d}{dt} Z = MZ + \frac{A_0}{\Lambda_0}[\rho(t)] + Q \tag{11a}$$

where

$$Z = \begin{bmatrix} C \\ P \end{bmatrix}$$

$$M = \begin{bmatrix} -\Delta & B \\ E & D \end{bmatrix}$$

$[\rho(t)] = \rho(t)$ in matrix form
C = column matrix $[C_i(t)]$ $\quad i = 1, 2 \cdots m$
P = column matrix $[a(t), \theta_j(t)]$ $\quad j = 1, 2 \cdots n$
$\Delta = m \cdot m$ matrix (see matrix I, below)
$B = m \cdot (n + 1)$ matrix (see matrix II, below)
$E = (n + 1) \cdot m$ matrix (see matrix III, below)
$D = (n + 1) \cdot (n + 1)$ matrix (see matrix IV, below)
Q = column matrix $[\zeta_i, \eta, \mu_j]$

When the delayed neutron precursors are not considered

$$\bar{\beta} = \beta_e^i = \lambda_i = 0$$

Eq. (11) reduces to:

$$\frac{d}{dt} P = D_0 P + \frac{A_0}{\Lambda_0}[\rho(t)] + Q_0 \tag{12}$$

where

$$D_0 = D, Q_0 = Q \quad \text{with} \quad \beta_i = \lambda_i = \rho_0 = 0$$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}$$

(I)

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

(III)

$$\begin{bmatrix} \beta_{e0}^1/\Lambda_0 & A_0 q_{11} & \cdots & A_0 q_{1n} \\ \beta_{e0}^2/\Lambda_0 & A_0 q_{21} & \cdots & A_0 q_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{e0}^m/\Lambda_0 & A_0 q_{m1} & \cdots & A_0 q_{mn} \end{bmatrix}$$

(II)

$$\begin{bmatrix} \frac{\rho_0 + \rho(t) - \beta_0}{\Lambda_0} & A_0 k_1 & \cdots & A_0 k_n \\ r_1 & \sigma_{11} & \cdots & \sigma_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ r_n & \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

(IV)

DELAYED NEUTRONS AND NONLINEAR STABILITY

In this section the effect of the delayed neutrons on the stability of the reactor is investigated by means of Liapunov's second method (*3*). It is shown that if the reduced reactor system [i.e., the reactor without delayed neutrons described by Eq. (12)] is stable for uniformly bounded variations of reactivity $\rho(t)$, then the actual reactor system with delayed neutrons, described by Eq. (11), is also stable under the same condition. The stability is characterized by the existence of a Liapunov function.

In order to proceed with the proof of the previous statement, the following assumptions, definitions, and assertions are necessary:

(*a*) Assume that the homogeneous reduced equation

$$(d/dt)P = D_0P \tag{13}$$

admits a Liapunov function $V(P, t)$. In other words, postulate that there is a region $R(G, \tau)$ in the multidimensional space (P, t), in which a definite positive function $V(P,t)$, with a definite negative time derivative exists. The function $V(P, t)$ is such that $\lim V(P, t) = 0$ as $\| P \| \to 0$, uniformly in t.

The existence of the Liapunov function $V(P, t)$ is equivalent to the assumption that the solution of equation (13) is asymptotically stable (*3*).

(*b*) Define the triangular matrix D_T by the transformation

$$D_T = T_0D_0T_0^{-1} \tag{14}$$

where T_0 is a uniformly bounded real matrix which possesses a uniformly bounded inverse. The existence of T_0 has been shown by Diliberto (*5*) and Perron (*3, 6*).

(*c*) Define the diagonal matrix H as

$$H = | H_s\delta_s^r | \qquad s = 1, 2 \cdots (n + 1) \tag{15}$$

where

$$H_s = G_s^{-2}(t, \tau) \int_t^\infty G_s^2(t', \tau)\, dt' \tag{16}$$

$$G_s(t, \tau) = \exp \int_\tau^t d_{ss}(t')\, dt' \qquad t > \tau \tag{17}$$

$$d_{ss} = \text{diagonal element of } D_T \tag{17a}$$

(*d*) Define the variable P_1 by the transformation

$$P_1 = ST_0P \tag{18}$$

where S is a diagonal matrix to be subsequently discussed.

(*e*) Admit with Perron (*3, 7*) that, if $\rho(t)$ is uniformly bounded, the necessary and sufficient condition for the solution of Eq. (12) to be stable is the existence of the Liapunov function $V(P, t)$ in a region $R(G, \tau)$ where $\| P \| < G$, $t > \tau$. The magnitude of $R(G, \tau)$ is determined by Q_0.

(*f*) Admit with Perron (*7*), Persidski (*8*), and Malkin (*9*) that the existence of the Liapunov function $V(P, t)$ implies the existence of two constants h_1 and h_2 such that

$$0 < h_1 < H_s(t, \tau) < h_2 \tag{19}$$

and the existence of a quadratic Liapunov function

$$L_1 = P_1^*(S^{-1*}HS^{-1})P_1 = P_1^*H_1P_1 \tag{20}$$

where X^* denotes the conjugate adjoint of X. A discussion of this problem can also be found in reference *3*.

(*g*) Introduce the change of variable $P_1 = ST_0P$ [Eq. (18)] into Eq. (11) and thus find

$$\frac{d}{dt}\begin{bmatrix} C \\ P_1 \end{bmatrix} = \begin{bmatrix} -\Delta & B_1 \\ E_1 & D_1 \end{bmatrix}\begin{bmatrix} C \\ P_1 \end{bmatrix} + T_0^{-1}S^{-1}\frac{A_0}{\Lambda_0}[\rho(t)] + T_0^{-1}S^{-1}Q \tag{21}$$

where

$$B_1 = BT_0^{-1}S^{-1} \qquad E_1 = ST_0E$$

$$D_1 = ST_0DT^{-1}S^{-1}$$

Notice that the application of the uniformly bounded transformations T_0, S does not change either the uniformly bounded character of $\rho(t)$ or the order of Q.

(*h*) Prove that the quadratic form

$$L = L_1 + C^*C \tag{22}$$

is a Liapunov function of the homogeneous part of equation (21) which is

$$\frac{d}{dt}\begin{bmatrix} C \\ P_1 \end{bmatrix} = \begin{bmatrix} -\Delta & \\ & E_1 \end{bmatrix}\begin{bmatrix} B_1 & C \\ D_1 & P_1 \end{bmatrix} \tag{23}$$

or

$$(d/dt)Z_1 = M_1Z_1 \tag{23a}$$

To this effect, consider the time derivative of L. Since C and P_1 are solutions of Eq. (23), after some elementary algebra, find that

$$\frac{dL}{dt} = \frac{dL_1}{dt} + Z_1^*[YM_{11} + M_{11}^*Y]Z_1 \tag{24}$$

where

$$Y = \begin{bmatrix} I & 0 \\ 0 & H_1 \end{bmatrix} \qquad I = m \cdot m \text{ unit matrix}$$

$$M_{11} = \begin{bmatrix} -\Delta & B_1 \\ E_1 & D_1 - D_{T1} \end{bmatrix} \qquad D_{T1} = SD_T S^{-1}$$

The sign of the first term on the right-hand side of equation (24) is always minus (see paragraph f, above):

$$(dL_1/dt) < 0 \tag{25}$$

The sign of the second term may be established as follows. Rewrite the second term as:

$$Z_1^*[YM_{11} + M_{11}^*Y]Z_1 = Z^*[YM_1^1 + M_1^{1*}Y]Z \tag{26}$$

where M_1^1 equals the matrix (V) shown below. The $(m + 1)\cdot(m + 1)$ submatrix of M_1^1, which corresponds to the variables $[c_i(t),\ a(t)]$, has characteristic values, solutions of the equation

$$\Lambda_0 p + \sum_i^m [\beta_{e0}^i p/(p + \lambda_i)] = 0 \tag{27}$$

Evidently, Eq. (27) admits $(m + 1)$ distinct roots, one equal to zero and m negative, because $\beta_{e0}^i > 0$. Since the characteristic roots of the submatrix are distinct, it is possible to perform an orthogonal transformation of $c_i(t)$ and $a(t)$ and reduce the submatrix to its diagonal form. If this orthogonal transformation is combined with a unit transformation of $\theta_j(t)$, the matrix M_1^1 is reduced to a triangular form M_1^T with all the main diagonal elements zero or negative. Thus, Eq. (26) becomes:

$$\begin{aligned} Z^*[YM_1^1 + M_1^{1*}Y]Z &= Z_2^*[YM_1^T + M_1^{T*}Y]Z_2 \\ &= 2Z_2^*[YM_1^T]Z_2 \end{aligned} \tag{28}$$

where Z_2 is the matrix resulting from Z when the orthogonal transformation of $[c_i(t),\ a(t)]$ and the unit transformation of $\theta_j(t)$, are applied. Following a procedure introduced by Malkin (*9*), the quadratic form (28) can be made to take the sign of the diagonal elements of M_1^T. This is achieved by choosing matrix S to be of the form

$$S = [\omega^{p-r}\delta_s^r] \qquad \omega > 0 \tag{29}$$

and properly adjusting the positive number ω. Consequently, the second term of Eq. (24) is always zero or negative, since it can be reduced to a negative quadratic form by orthogonal and similarity transformations.

$$Z_1^*[YM_{11} + M_{11}^*Y]Z_1 \leqq 0 \tag{30}$$

The combination of inequalities (29) and (30) yields

$$(dL/dt) < 0 \tag{31}$$

which establishes the fact that L is a Liapunov function.

Now, on the basis of assumptions and assertions (a) through (h) it is easy to ascertain the statement about the effect of delayed neutrons on stability. Since all the transformations used in the preceding discussion are, geometrically speaking, rotations or dilatations, the Liapunov character of the quadratic form L is definite and independent of the coordinate system Z_i. Therefore, it is concluded that the homogeneous part of Eq. (11) admits a Liapunov function. According to Perron this implies that there is a region $R(Z, \tau)$, in the multidimensional space (Z, t), in which the solution of Eq. (11) is stable. Of course, the magnitude of $R(Z, \tau)$ is determined by the second-order terms included in Q.

In summary, the following theorem has been proved: If the reduced system of kinetics equations, without delayed neutrons, is stable, then the complete system, which includes delayed neutrons, is

$$M_1^1 = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{-\lambda_1 \quad 0 \quad \cdots \quad -\lambda_m \quad \textstyle\sum_i^m \beta_{e0}^i/\Lambda_0}}^{m+1} & \overbrace{\hphantom{\Lambda_0 q_{11} \quad \cdots \quad \Lambda_0 q_{1n}}}^{n} \end{array} \\ \left[\begin{array}{ccccc|ccc} -\lambda_1 & 0 & \cdots & 0 & \beta_{e0}^1/\Lambda_0 & \Lambda_0 q_{11} & \cdots & \Lambda_0 q_{1n} \\ 0 & -\lambda_2 & \cdots & 0 & \beta_{e0}^2/\Lambda_0 & \Lambda_0 q_{21} & \cdots & \Lambda_0 q_{2n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & -\lambda_m & \beta_{e0}^m/\Lambda_0 & \Lambda_0 q_{m1} & \cdots & \Lambda_0 q_{mn} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m & \sum_i^m \beta_{e0}^i/\Lambda_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \\ a \end{array}}\right\} m+1 \\ \left.\vphantom{a}\right\} n \end{array}$$

(V)

also stable. Furthermore, it can be easily shown that asymptotic stability of the reduced system implies the asymptotic stability of the complete system.

ILLUSTRATIVE EXAMPLE

In this section the value of the derived theorem is illustrated by means of an example. Consider a heterogeneous reactor with two temperature coefficients of reactivity. The complete system of equations describing the dynamics of this reactor is *(10)*

$$\frac{dA(t)}{dt} = -[r_1\theta_1(t) + r_2\theta_2(t)]A(t) + \sum_i^6 \lambda_i C_i(t) \quad (32)$$

$$\epsilon_1 \frac{d\theta_1(t)}{dt} = y_1[A(t) - A_0] - X[\theta_1(t) - \theta_2(t)] \quad (33)$$

$$\epsilon_2 \frac{d\theta_2(t)}{dt} = y_2[A(t) - A_0] - X[\theta_2(t) - \theta_1(t)] \quad (34)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_e^{\,i}}{\Lambda} A(t) - \lambda_i C_i(t) \quad i = 1, 2 \cdots 6 \quad (35)$$

where

$A(t)$	Reactor power
r_1, r_2	Ratio of temperature coefficient of reactivity to neutron lifetime
X	Thermal conductivity
y_1, y_2	Fractional power generated in medium 1 or 2, respectively ($y_1 + y_2 = 1$)
ϵ_1, ϵ_2	Heat capacity
$\theta_1(t)$, $\theta_2(t)$	Temperature increment

The system of the nine equations (32) to (35) is rather involved to analyze. The reduced system, however, of the three equations (32) to (34) without delayed neutrons, after some elementary algebraic operations, yields *(10)*

$$\frac{d}{dt}L = \frac{d}{dt}\left[\frac{A(t)}{A_0} - 1 - \ln\frac{A(t)}{A_0} + \sigma_1[\epsilon_1\theta_1(t) + \epsilon_2\theta_2(t)]^2 + \sigma_2[\theta_1(t) - \theta_2(t)]^2\right] = -2\sigma_2 t_0[\theta_1(t) - \theta_2(t)]^2 \quad (36)$$

where

$$\sigma_1 = \frac{1}{2A_0}\cdot\frac{r_1 + r_2}{\epsilon_1 + \epsilon_2}$$

$$\sigma_2 = \frac{1}{2A_0}\cdot\frac{r_1\epsilon_2 - r_2\epsilon_1}{\epsilon_1 + \epsilon_2}\cdot\frac{1}{(y_1/\epsilon_1) - (y_2/\epsilon_2)}$$

$$t_0 = X\frac{\epsilon_1 + \epsilon_2}{\epsilon_1\epsilon_2}$$

The function L is a Liapunov function if

$$\sigma_1 > 0 \quad \text{and} \quad \sigma_2 > 0 \quad (37)$$

Therefore, it can be immediately concluded that the solutions of the complete system of Eqs. (32) to (35) are stable, and in fact asymptotically stable, if conditions (37) are true.

DISCUSSION

It has been shown, quite generally, that if a reactor without delayed neutrons is stable for uniformly bounded changes of reactivity, then the same reactor with delayed neutrons will also be stable. This assertion is of great value to the study of the problem of stability of nonlinear reactor dynamics. It reduces the number of kinetic equations by m and thus greatly simplifies the mathematical complexity of the problem. It should be pointed out, however, that if stability is not affected by the delayed neutrons, the actual solutions of the kinetic equations are modified appreciably, as it is well known, even in the case of simple linear reactor models.

The reactor dynamics have been assumed describable by a set of differential equations. Strictly speaking, this is not completely general, because there are reactor designs where integral operators, not reducible to differential equations, are introduced in the feedback loops. However, the dynamics of such reactors can be approximated as closely as desired by a series of differential equations. Therefore the previous conclusions are applicable since the derivations of this paper are independent of the number of equations assumed.

A comment is necessary on the size of the region $R(Z, \tau)$ in which the derived stability criterion is valid. Many studies of stability in the field of reactor kinetics introduce Liapunov functions with unbounded domains of validity. This feature allows the consideration of starting points of trajectories anywhere in the space (Z, t) but at the same time reduces the scope of the method. The advantage of considering any starting point is likely to be often offset by the fact that the kinetic equations are not valid for large displacements. The region $R(Z, \tau)$ introduces a restriction on the size of displacements, which in most of the practical cases is not as restrictive as the conditions imposed by the validity of the basic equations. The size of $R(Z, \tau)$ can be determined in each specific case and its importance can be assessed by comparison with the domain of validity of the nonlinear kinetic equations.

Finally, it should be recognized that the assumed existence of a Liapunov function for the reduced system of kinetics equations is more restrictive than the requirement of mere stability. It can be shown that if all systems having a Liapunov function are stable, not all stable systems possess a Liapunov function (*3*). However, this fact rests on very particular mathematical hypotheses and therefore may be overlooked when discussing physical problems.

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