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Section V

The Measurement of Dynamic Characteristics of Nonlinear Systems.

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by

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1. INTRODUCTION

The question of understanding the dynamics of physical systems, from an engineering standpoint, is often approached by means of the powerful technique of input-output type measurements. For example, the dynamics of a nuclear reactor are explored by varying the position of a control rod in a predetermined or statistically controlled manner and measuring the corresponding variations of the power output. A similar approach is directly applicable to a large variety of physical systems even though the input-output characterization and nature may be entirely different.

There are two mathematical problems that are related to the interpretation and use of input-output data. The first problem is one of analysis. More precisely, given a set of input-output measurements, the question is how can one extract from them enough information to derive a mathematical model for the system ? This model, of course, must be both the "best" approximation to the functional relationship between the given input and output as well as be capable of predicting the behaviour of the output of the physical system for any other kind of input. The second problem is somewhat similar but one of synthesis. Specifically, given a certain input and a disirable output the question is what physical components, of known input-output characteristics, should be used to synthesize a system which produces a close approximant of the desired output when excited by the given input ?

Finally, there is a third problem which is physical in nature. More precisely, having analyzed a set of input-output data and having established a mathematical model, what can one say about the physical processes that characterize the system ?

These problems can be more or less easily solved if the functional relationship between the input and the output is linear. Physical systems, however, in general and nuclear systems in particular are characterized by nonlinear relationships and the problems of analysis, synthesis and physical interpretation are much more involved.

The purpose of these notes is to review some of the relatively recent developments in nonlinear theory pertinent to the previously stated problems, to indicate how this theory can be used in the analysis of dynamic measurements in nuclear reactors and to present an interpretation of reactor transfer function measurements.

The following section is devoted to the mathematical description of stationary, physically realizable, nonlinear systems as it has been developed by Wiener (1) and others (2-3). The third section discusses various procedures by means of which the theory can be implemented in analyzing data on nuclear reactors. The fourth section gives a series of applications and the last section presents a general discussion of the possibilities and limitations of the method.

2. THE FUNCTIONAL REPRESENTATION OF SYSTEMS

2.1 General Remarks.

This section is concerned with the mathematics used in the problem of analysis and synthesis of stationary, physically realizable*, nonlinear systems which from an engineering standpoint of view can be visualized as in Figure 1. The discussion is general without reference to any particular physical system, input or output. Only one input and one output are considered for mathematical expediency. The formalism can be readily extended to any number of inputs and/or outputs (4).

2.2 The Functional Expansion of Input-Output Relations.

The implication of the representation of Figure 1 is that the input x(t) enters the system, is processed by the system and appears as an output y(t). Therefore, measurements of the input and the corresponding output contain the information about the dynamics of the system. For the types of systems under consideration, this information can be stated analytically in terms of a functional:

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* Realizable means here that the output of the system depends only on the infinite past history of the input and not on its future values.

$$y(t) = F(x(t-\tau)) \qquad \tau \leq t \qquad (1)$$

The physical meaning of the analytical statement (1) is that the present value of the output of a stationary, physical system is uniquely determined only by the past history of the input. The exact character of the dependence, of course, is determined by the specific system on which the input-output measurements are taken.

The functional F can be expanded into a series of functionals, a form which is more suggestive of the approach to the analysis and synthesis problems. To this effect, consider an arbitrary complete set of orthonormal functions $\Phi_n(t)$ defined over the range 0 - T. The range T may be finite or infinite depending on how far back the system remembers the past history of the input. Expand the past values of the input $x(t-\tau)$ into a Fourier series in terms of the members of the complete set $\Phi_n(t)$. In other words:

$$x(t-\tau) = \sum_{i=1}^{\infty} u_i(t)\, \Phi_i(\tau) \qquad (2)$$

$$u_i(t) = \int_0^T x(t-\tau)\, \Phi_i(\tau) d\tau \qquad (3)$$

Equations (2) and (3) have several important implications. The Fourier coefficients $u_i(t)$ are functions of the present time only and constitute an equivalent representation of the past of the input. The necessary and sufficient condition for the expansion to be meaningful is that the input be class L^2 (square integrable) over the range of interest. In principle, an infinite number of Fourier coefficients is required to represent $x(t-\tau)$. However, in case the expansion is truncated at $i = N < \infty$, the input is approximated in the least mean square error sense. Finally, if each member $\Phi_i(t)$ is visualized as the impulse response of a linear system, the coefficients $u_i(t)$ are the outputs of systems like the one shown in fig. 2.

Next, replace $x(t-\tau)$ in eq. (1) by the equivalent set $u_i(t)$ and find:

$$y(t) = F(u_1(t), u_2(t) \ldots u_k(t) \ldots) \qquad (4)$$

Notice that now the output y(t) is a nonlinear functional of the Fourier coefficients $u_i(t)$. Eq. (4) has a simple "black box" representation as shown in fig. 3. The input is fed in parallel into an infinite set of linear systems. The outputs of these systems are combined in a nonlinear manner to produce the output y(t). The conceptual difference between the systems of figures 1 and 3 is that in the first case the nonlinear system as a whole has memory while in the latter case the memory is attributed to an infinite set of linear systems and the nonlinearities to a memoryless functional operation.

Temporarily disregarding the necessary requirements, expand eq. (4) into a power series:

$$y(t) = a_o + \sum_i a_i u_i(t) + \sum_{i,j} a_{ij} u_i(t)\, u_j(t) + \ldots + \sum_{i,j..k} a_{ij..k} u_i(t) .. u_k(t) + \ldots \qquad (5)$$

The coefficients $a_i, a_{ij} .. a_{ij..k}$ are characteristic of the nonlinear system under consideration and differ from system to system. This becomes even more evident if eq. (3) is replaced into eq. (5) to yield:

$$y(t) = h_o + \int_0^T h_1(\tau)\, x(t-\tau) d\tau + \int_0^T \int_0^T h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 + \ldots + \int_0^T \int_0^T \ldots \int_0^T h_k(\tau_1, \tau_2 \ldots \tau_k) x(t-\tau_1) \ldots x(t-\tau_k) d\tau_1 \ldots d\tau_k + \ldots \qquad (6)$$

where

$$h_o = a_o$$

$$h_1(\tau) = a_1 \Phi_1(\tau) + a_2 \Phi_2(\tau) + \ldots = \sum_i a_i \Phi_i(\tau)$$

$$h_2(\tau_1,\tau_2) = \sum_{i,j} a_i \Phi_i(\tau_1)\ \Phi_j(\tau_2)$$

....

$$h_k(\tau_1,\tau_2 \ldots \tau_k) = \sum_{i,j,k} a_{ij\ldots k} \Phi_i(\tau_1) \ldots \Phi_k(\tau_k)$$

Equation (6) is the functional expansion that was sought. In principle, the nonlinear system is now characterized by an infinite set of kernels $h_k(\tau_1,\tau_2 \ldots \tau_k)$ and has a black box equivalent as shown in figure 4. The contribution from each kernel to the output is derived by a generalized convolution operation. Notice that if the system is linear ($h_k = 0$, $k \geq 2$) the functional expansion reduces to the well known linear convolution.

Functionals of the type appearing in eq. (6) have been studied by Volterra (2). Wiener (1) used the functional expansion to investigate nonlinear electrical circuit problems. Other authors investigated different properties of functionals (4-6) and developed a systematic algebra and multi-Laplace transformation theory for a system or combination of systems represented by functional expansions (6). The findings of these authors will not be discussed here. The interested reader is referred to reference (3) for an excellent summary.

Equation (6), as it stands now, is not always convenient for analysis of input-output measurements. However, it is suggestive of a similar expansion which is more suitable for both theoretical and experimental studies. Specifically, if the expansion were in terms of an orthonormal set of functionals, a number of advantages are evident (7):

a. If the expansion is truncated at a finite number of functionals, the input is approximated in the least mean square error sense.

b. Each member of the expansion is linearly independent of the others and the number of functionals considered.

c. Each member of the expansion can be determined by use of the orthogonality relationship.

These advantages have been appreciated by Wiener (1) who rigorously developed a canonical expansion for nonlinear systems excited by gaussian, white noise inputs. Wiener's theory is summarized in the next section.

2.3 Wiener's Canonical Representation of Nonlinear Systems.

Wiener's rigorous theory of nonlinear systems is decribed in his monograph "Nonlinear Problems in Random Theory" (1). However, since this theory is not well known yet, the major results are reviewed here in some detail without any proofs. For the latter, the interested reader is referred to Wiener's monograph.

The primary objective of Wiener's approach to the analysis and synthesis problems of nonlinear systems is to derive a canonical functional expansion for a given set of input-output data.

To this effect Wiener first considers an ensemble of gaussianly distributed, white noise signals as shown in fig. 5. He shows that there is a finite and well defined probability that a member of the ensemble approximates arbitrarily close, in the least mean square error sense, any time function, provided that the latter is L^2. In this sense then, a gaussian white noise signal may be considered as a well defined function of time which Wiener calls the stochastic function x(t).

In view of these remarks it is evident that the stochastic function is the best input to use to probe the dynamics of a physical system since it can simulate almost any conceivable practical input.

Next, Wiener consideres an arbitrary, symmetrical* kernel

$K_n(\tau_1, \tau_2 \dots \tau_n)$ and the functional

$$f_n(t) = \int \dots \int K_n(\tau_1, \tau_2 \dots \tau_n) x(t-\tau_1) \dots x(t-\tau_n) d\tau_1 \dots d\tau_n \qquad (7)$$

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* The requirement of symmetry does not limit the generality of the discussion. If K_n is not symmetric, it can by symmetrized by considering the sum

$$K^* (\tau_1, \tau_2 \dots \tau_n) = \frac{1}{n!} \sum_n K_n(\tau_i, \tau_j \dots \tau_k)$$

taken over all possible permutations of the τ_i's $(i \le n)$

and he shows readily that for n = odd:

$$\int f_n(t)dt = \int dt \int\!\!..\!\!\int K_n(\tau_1, \tau_2 \dots \tau_n) x(t-\tau_1)\dots x(t-\tau_n) d\tau_1 \dots d\tau_n = 0 \tag{8}$$

and for n = even (n = 2m):

$$\int f_n(t)dt = \int dt \int\!\!..\!\!\int K_n(\tau_1, \tau_2 \dots \tau_n) x(t-\tau_1)..x(t-\tau_n) d\tau_1 .. d\tau_n =$$

$$= (2m-1)(2m-3)..(1)\int\!\!..\!\!\int K_n(\tau_1, \tau_1 \dots \tau_m, \tau_m) d\tau_1 d\tau_2 .. d\tau_m \tag{9}$$

where the integrations are from $-\infty$ to ∞.

On the basis of eqs. (8) and (9) he then proceeds and establishes an orthogonal set of functional polynomials of arbitrary kernels K_n and the stochastic function x(t). These orthogonal functional polynomials are:

$$G_o = \text{constant}$$

$$G_1(K_1, x, t) = \int K_1(\tau) x(t-\tau) d\tau$$

$$G_2(K_2, x, t) = \int\int K_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 - \int K_2(\tau, \tau) d\tau$$

$$G_3(K_3, x, t) = \int\int\int K_3(\tau_1, \tau_2, \tau_3) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) d\tau_1 d\tau_2 d\tau_3 -$$

$$-3\int\int K_3(\tau_1, \tau_1, \tau) x(t-\tau) d\tau_1 d\tau$$

$$G_n(K_n, x, t) = \sum_{\nu=o}^{[\frac{n}{2}]} a_{n-2\nu}^{(n)} \int \dots \int K_n(\tau_1, \tau_2 \dots \tau_n) x(t-\tau_1) \dots (t-\tau_{n-2\nu}) \cdot$$

$$\cdot \delta(\tau_{n-2\nu+1} - \tau_{n-2\nu+2}) \dots \delta(\tau_{n-1} - \tau_n) d\tau_1 d\tau_2 \dots d\tau_n$$

where $[\frac{n}{2}] = \frac{n}{2}$ for n = even and $\frac{n-1}{2}$ for n odd and

$$a_{n-2\nu}^{(n)} = \frac{(-1)^{\nu} n!}{2^{\nu}(n-2\nu)!\nu!}$$

The functional G_1 is orthogonal to all constants. The functional G_2 is orthogonal to all constants and all functionals G_1. The functional G_n is orthogonal to all functionals $G_k (k < n)$ etc. The functionals can but need not be normalized through the normalization condition:

$$n! \int \dots \int K_n^2(\tau_1, \tau_2 \dots \tau_n) d\tau_1 d\tau_2 \dots d\tau_n = 1 \qquad (10)$$

The derivation of the orthogonal functionals G_n is straightforward and follows the standard procedure of formation of orthogonal functions (7). The orthogonality is defined over the time variable:

$$\int G_1(K_1, x, t) G_m(K_m, x, t) dt = \delta_{1m} \qquad (11)$$

$$\int G_n(K_n, x, t)\; G_n(L_n, x, t) dt = n! \int \dots \int K_n L_n d\tau_1 \dots d\tau_n \qquad (12)$$

It is easily shown that the functionals G_n form a complete set. Therefore, any function of time which is L^2 can be expanded in a series of G_n functionals in a unique way:

$$y(t) = \lim_{N \to \infty} [\, G_0 + G_1 (K_1, x, t) + \dots + G_N(K_N, x, t)] \qquad (13)$$

Again this is just like ordinary orthogonal development except that here the development is in terms of functionals.

If y(t) in eq. (13) is the output of a nonlinear system whose input is a gaussian, white noise signal then eq. (13) is the canonical representation of the system and the kernels K_n are the characteristic kernels of the system. This representation is similar to that of eq. (6). The important difference is that it provides a definite technique for the measurement of the kernels K_n. The technique is discussed in detail in section 3. Another difference is that the functionals G_k are generalized polynomials of the input function x(t) instead of depending on the first, second...nth power of the input as in the case of the expansion of eq. (6).

This completes the review of Wiener's theory. Before proceeding with the discussion of its usefulness in problems of analysis and synthesis of nonlinear systems it is interesting to consider orthogonal expansions for non-gaussian inputs.

2.4 Orthogonal Expansions of Nonlinear Functionals.

The question of interest here is whether it is possible to expand a nonlinear functional of the type indicated by eq. (1) in terms of orthogonal functionals when the input is not a stochastic function. It is shown below that this is at least conceptually possible.

To this effect, consider a periodic input x(t) whose period is T. Assume that there is a set of polynomials

$$p_o = \text{const},\ p_1(x(t-\tau_1)),\ p_2(x(t-\tau_1),x(t-\tau_2))\ldots \qquad (14)$$

$$\ldots p_n(x(t-\tau_1),\ldots x(t-\tau_n))$$

determined by the following conditions:

a. p_k is a polynomial of x of degree k (k = integer)

b. The polynomials p_k are orthonormal over the period T, that is

$$\frac{1}{T}\int_0^T p_m p_n dt = \begin{cases} \delta_{mn} \\ \text{or combinations of delta functions} \end{cases} \qquad (15)$$

where δ_{mn} the Kronecker delta

c. The polynomials $p_k(x)$ form a complete set.

Under these conditions the functionals

$$f_m = \int_0^T d\tau_1 \ldots \int_0^T d\tau_m H_m(\tau_1, \tau_2, \ldots \tau_m) p_m \quad \text{and}$$

$$f_n = \underbrace{\int_0^T d\tau_1 \ \ldots \int_0^T}_{n} d\tau_n H_n(\tau_1, \tau_2, \ldots \tau_n) p_n \tag{16}$$

are orthogonal regardless of the value of the kernels

$H_m(\tau_1 \ldots \tau_m)$ and $H_n(\tau_1 \ldots \tau_n)$. Indeed

$$\int_0^T f_m f_n dt = \underbrace{\int_0^T d\tau_1 \ldots \int_0^T d\tau_m}_{m} \underbrace{\int_0^T d\tau_1' \ldots \int_0^T d\tau_n'}_{n} H_m(\tau_1 \ldots \tau_m) H_n(\tau_1' \ldots \tau_n') \int_0^T dt p_m p_n =$$

$$= \delta_{mn} \int_0^T d\tau_1 \ldots \int_0^T d\tau_m \int_0^T d\tau_1' \ \ldots \int_0^T d\tau_n' H_m H_n \tag{17}$$

Consequently, following a procedure similar to that indicated in sections 2.2 and 2.3 it can be readily established that eq. (1) can be written as

$$y(t) = H_0 + \int_0^T H(\tau_1) p_1 d\tau_1 + \int_0^T d\tau_1 \int_0^T d\tau_2 H_2(\tau_1, \tau_2) p_2 + \ldots \tag{18}$$

where the different functionals are orthogonal.

If the establishment of the complete set p_k were feasible, the orthogonal expansion (18) would be more practical than (13) because the range of orthogonality is finite and not infinite as in the case where the input is the stochastic function.

3. THE MEASUREMENT OF THE KERNELS OF ORTHOGONAL EXPANSIONS.

3.1 The Measurement of the Wiener Kernels.

The value of Wiener's canonical representation of nonlinear systems does not only lie in the fact that the stochastic function can simulate any conceivable practical input but also that it is suggestive of a very simple procedure for the measurement of the kernels K_n. To see this clearly consider the experimental set up shown in fig. 6. In other words, assume that the stochastic function is fed simultaneously into the input of the physical system under testing and into the input of a known system whose characteristic is the nth order Wiener functional $G_n(L_n,x,t)$. In addition, assume that the outputs of the two systems are multiplied and integrated over a long time. If the output of the system under testing is represented by a canonical Wiener expansion with kernels K_n, then the output of the integrator is (see eqs. (11) and (12)):

$$I = n! \underbrace{\int \cdots \int}_{n} K_n L_n \, d\tau_1 d\tau_2 \ldots d\tau_n \tag{19}$$

That is the output of the integrator depends only on the K_n kernel of the unknown nonlinear system. Actually, if the known system is as shown in fig. 7, then L_n is just the product of n delta functions and the output of the integrator is just

$$I = n! \; K_n(\tau_1', \tau_2', \tau_3' \ldots \tau_n') \tag{20}$$

provided that $\tau_1' \neq \tau_2' \neq \ldots \neq \tau_n'$. Consequently, by repeating the experiment for different delays τ_i', the entire range of values of the kernel K_n can be measured.

In summary, Wiener's canonical representation permits the measurement of all the kernels K_n of a nonlinear system by means of a generalized cross correlation technique. Specifically, the zeroth Wiener functional is just the average value of the output, the kernel of the first functional can be measured by the experimental set up of fig. 8, the kernel of the second functional can be measured by the experimental set up of fig. 9 and so on. It is understood that the experimental procedures described above can be performed either on line or by means of a digital computer.

A major disadvantage of Wiener's theory and experimental procedures is the requirement for extremely long times of integration. Theoretically this time should be infinite. If the time of integration is very short, then the input does not behave like a gaussian white noise signal and the resulting errors can be very large.

An experimental procedure similar to that described for the measurement of Wiener's kernels, can be readily prescribed for the orthogonal expansion of section 2.4, if a signal with the aforementioned properties were available. At the present such a signal is not available. Work, however, is currently under way and it is hoped that soon such a signal will be designed for testing of nonlinear systems.

Inspite of its shortcomings Wiener's canonical representation as well as the visualization of system dynamics in terms of functionals are useful tools for a variety of theoretical and practical investigations.

4. APPLICATIONS

The purpose of this section is to illustrate the usefulness of functional expansions by means of some specific nuclear reactor problems.

4.1 The Describing Function of a Nuclear Reactor.

The dynamic equations of a nuclear reactor are either linear with time dependent coefficients or nonlinear. If the reactor is excited by a sinusoidal input then the steady state output (be it power, temperature etc.) contains components both of the same frequency as the input and of higher frequencies. The describing function (8) of the reactor is defined as the complex ratio of the amplitude of the fundamental component of the output to the amplitude of the sinusoidal input. For linear systems, the describing function reduces to the well known notion of the transfer function. In general, the describing function is a function of both the frequency and the amplitude of the input.

Theoretical derivations of the reactor describing function have already been presented (9-12). However, in all these derivations mathematical assumptions were made which were not implemented. The consequence of this is that the results are slightly in error. Here the describing function is derived from a functional expansion of the reactor dynamics and particular attention is paid to mathematical rigor.

Consider the reactor kinetics equations at low power (no feedback).

$$\frac{d\Phi}{dt} = \frac{\rho_1 - \beta}{\Lambda} \Phi + \sum_{i=1}^{m} \lambda_i C_i + Q_o \tag{21}$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} \Phi - \lambda_i C_i \tag{22}$$

Assume that at time t = 0, the initial values are Φ_o, ρ_o, C_{io} and Q_o. Suppose that after t = 0 the reactivity is $\rho_1 = \rho_o + \rho(t)$, and that all variables are written as the sum of the steady state plus the time dependent increments. Thus, eqs.(21) and (22) can be written as:

$$\frac{d\Phi}{dt} = \frac{\rho_o - \beta}{\Lambda} \Phi + \frac{\Phi_o}{\Lambda} \rho + \frac{1}{\Lambda} \rho \Phi + \sum_{i}^{m} \lambda_i C_i \tag{23}$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} \Phi - \lambda_i C_i \tag{24}$$

The Laplace transform of eqs. (23-24) is

$$s\bar{\Phi} = \frac{\rho_o - \beta}{\Lambda} \bar{\Phi} + \frac{\Phi_o}{\Lambda} \bar{\rho} + \frac{1}{\Lambda} \overline{\rho\Phi} + \sum_{i}^{m} \lambda_i C_i \tag{25}$$

$$s\bar{C}_i = \frac{\beta_i}{\Lambda} \bar{\Phi} - \lambda_i \bar{C}_i \tag{26}$$

or

$$\frac{\bar{\Phi}}{\Phi_o} = \bar{\Phi}_1 = \bar{\rho} G(s) + \overline{\rho\Phi_1} G(s) \tag{27}$$

where

$$G(s) = \frac{\beta}{\left[\Lambda s + \sum_{i}^{m} \frac{\beta_i s}{(s+\lambda_i)} - \rho_o \right]} \tag{28}$$

g(t) = inverse of G(s)
and ρ is measured in dollars. Notice that G(s) is the transfer function of the reactor with an initial reactivity ρ_o. The inverse transform of (27) is

$$\Phi_1(t) = \int_0^t g(\tau)\rho(t-\tau)d\tau + \int_0^t g(\tau)\Phi_1(t-\tau)\rho(t-\tau)d\tau \qquad (29)$$

Equation (29) is a Volterra integral equation of the first kind. If the equations of the reactor were linear, the second term on the right hand side of (29) would be missing, and the neutron flux would be the convolution of the system function and the input ρ.

The solution of eq. (29) is given by Volterra (2). It is

$$\Phi_1(t) = \int_0^\infty d\tau g(\tau)\rho(t-\tau) + \int_0^\infty d\tau \int_0^\infty d\tau_1 g(\tau_1)g(t-\tau)\rho(\tau-\tau_1)\rho(\tau) +$$

$$+ \int_0^\infty d\tau \int_0^\infty d\tau_1 g(\tau_1)\rho(t-\tau_1) \int_\tau^t d\tau_2 \Phi(\tau)g(\tau_2-\tau)\rho(\tau_2)g(t-\tau_2) + \dots \qquad (30)$$

If the integrands are uniformly convergent, namely if

$$\int_0^\infty |g(t)| \, dt < M < \infty \qquad (31)$$

or the reactor is subcritical ($\rho_o < o$), then the order of integration of the various terms of (30) can be interchanged and the result can be written, after some simple manipulations, as:

$$\Phi_1(t) = \int_0^\infty d\tau g(\tau)\rho(t-\tau) + \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 g(\tau_1)g(\tau_2)\rho(t-\tau_1)\rho(t-\tau_1-\tau_2) +$$

$$(32)$$

$$+ \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 g(\tau_1)g(\tau_2)g(\tau_3)\rho(t-\tau_1)\rho(t-\tau_1-\tau_2)\rho(t-\tau_1-\tau_2-\tau_3) + \dots$$

or

$$\Phi_1(t) = \int_0^\infty d\tau g(\tau)\rho(t-\tau) + \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 g(\tau_1)g(\tau_2-\tau_1)\rho(t-\tau_1)\rho(t-\tau_2) +$$

$$+\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 g(\tau_1)g(\tau_2-\tau_1)g(\tau_3-\tau_2)\rho(t-\tau_1)\rho(t-\tau_2)\rho(t-\tau_3)+\ldots \quad (33)$$

Equation (33) is a functional expansion similar to eq. (6). The important difference here is that the kernels are given explicitly in terms of the kernel of the linearized system ($g(t)$). Also the derivation of (32) or (33) brought out the specific requirement (31) as well as the procedure used to reduce a set of differential equations into an integral equation. All these points are important when one attempts to assign physical significance to input-output interrelationships.

The reactor describing function can be readily derived from eq. (32). Indded suppose that:

$$p(t) = A_1 \sin \omega t = k_1 e^{j\omega t} + k_1^* e^{-j\omega t} \quad (34)$$

$$k_1 = A_1/2j \qquad k_1^* = \text{conjugate of } k_1$$

Define: $$\int_0^\infty g(t)dt = M = -\frac{\beta}{\rho_o} \quad (35)$$

$$\int_0^\infty g(t)e^{-jn\omega t} = G_n \quad (36)$$

Replace (34) in (32) and thus find

$$\Phi_1(t) = k_1 e^{j\omega t} G_1 + k_1^* e^{-j\omega t} G_1^* +$$

$$+ k_1 k_1^* (MG_1 + MG_1^*) + k_1^2 e^{2j\omega t} G_1 G_2 + k_1^{*2} e^{-2j\omega t} G_1^* G_2^* +$$

$$+ k_1^2 k_1^* e^{j\omega t}(MG_1 G_1^* + MG_1^2 + G_1^2 G_2) +$$

$$+ k_1^{*2} k_1 e^{-j\omega t}(MG_1^* G_1 + MG_1^{*2} + G_1^{*2} G_2^*) +$$

$$+ k_1^3 e^{3j\omega t} G_1 G_2 G_3 + k_1^{*3} e^{-3j\omega t} G_1^* G_2^* G_3^* + \ldots \quad (37)$$

Combine all the coefficients of $e^{j\omega t}$ and divide by k_1 to find that the describing function is

$$D(\omega,k_1) = G_1\Big[1 + k_1 k_1^* \; (2M\text{Re}G_1 + G_1 G_2) + + k_1^2 k_1^{*2}\Big\{(2M\text{Re}G_1)^2 + 2M(G_1 G_2 \text{Re}G_1 + \text{Re}G_1^2 G_2) + G_1 G_2^2 (G_1 + G_3)\Big\} + \ldots\Big] \quad (38)$$

The meaning of eq.(38) is that if the reactivity amplitude of oscillation is large (larger than 10¢) oscillation tests do not yield the transfer function G(s) but the describing function D(ω,k).

For further discussion of the concept of the describing function the reader is referred to references (9-12).

The case of the describing function of a nuclear reactor with linear feedback, i.e.

$$\rho(t) = \int_0^t f(t-\tau)\Phi(\tau)d\tau \quad (39)$$

can be derived in a similar manner but the derivations and mathematical manipulation are much more involved.

4.2 Comparison of Oscillation, Autocorrelation and Crosscorrelation Tests.

The transfer function of a nuclear reactor system is usually measured by means of oscillation, autocorrelation or crosscorrelation tests. If the reactor system were linear, all three types of measurements would yield identical results*. As it is well known a reactor system is nonlinear and the question arises as to the exact meaning of

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* Of course identical results are also derived when the amplitude of the perturbation of all variables is small compared to their steady state values.

the different types of measurements mentioned above.

This question can be elegantly answered if the reactor dynamics is represented by Volterra or Wiener type functionals.

Consider first the case of oscillation tests. If the system is represented by Volterra functionals (eq.(6) or (32)), oscillation tests yield in general the describing function of the system, as already indicated in section 4.1. The describing function is essentially a combination of contributions from all the odd order kernels. Similarly, if the system is represented by a Wiener type functional expansion and the input is a sinusoid instead of a stochastic function, it is easily seen that oscillation tests yield a result dependent on all the odd order kernels.*

Next consider the case of autocorrelation tests. Here, it is ordinarily assumed that the input is a gaussian white noise signal (which may not necessarily be true). Thus, Wiener's canonical expansion is appropriate for the output. The autocorrelation of the output is

$$\overline{y(t)y(t+\tau)} = \sum_n \overline{G_n(K_n,x,t)G_n(K_n,x,t+\tau)} = \tag{40}$$

$$= \sum_n \underbrace{\int\cdots\int}_{2n} K_n(\tau_1,\ldots\tau_n)K_n(\tau_{n+1},\ldots\tau_{2n})\overline{x(t-\tau_1)\ldots x(t-\tau_n)\cdot}$$

$$\overline{\cdot\, x(t-\tau_{n+1}+\tau)\ldots x(t-\tau_{2n}+\tau)}\,d\tau_1\ldots d\tau_{2n}$$

The meaning of eq. (40) is that the autocorrelation of the reactor power, when the reactor is excited by a gaussian white noise signal, is related to all the kernels of the system. Therefore, in general the result of autocorrelation tests is different from the result of oscillation tests.

Incidentally, note that in presenting Wiener's formalism it is assumed that the power density of the stochastic function is unity. Had

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* It is assumed that during oscillation tests, only the fundamental component of the output is measured.

it been assumed different, the autocorrelation (40) would be a power series of the power density. Thus, if the power density is small the higher order terms would be negligible and the autocorrelation would be proportional to the power density of the transfer function.

Finally, consider the case of crosscorrelation tests. If the reactor is excited by a gaussian white noise signal then crosscorrelation of the input and the output (see section 3) would yield the first order kernel, namely the best (in the least mean square error sense) linear approximation to the nonlinear system. Higher order crosscorrelations would, of course, result in the higher order kernels. Consequently, the best method of measuring the reactor transfer function, without any limitations on the amplitude of input, is by means of a gaussian white noise input and a crosscorrelation of the input and the output. Of course, the gaussian signal has its headaches but hopefully signals will become available which have similar properties over finite time intervals.

The measurement of the transfer function is important because many sufficient criteria of stability are based on the stability of the linearized model of the nonlinear system. This topic is discussed in another section.

5. DISCUSSION

The functional representation of the dynamic behaviour of a physically realizable system is a very useful tool for the study of the transient behaviour of nuclear reactor systems. The major advantage is that when the functionals are orthogonal one can measure all the kernels of the functional expansion by means of simple procedures, either on line or with the aid of a digital computer. The orthogonal expansion is possible when the input is a gaussian white noise signal.

The disadvantage of the gaussian input and Wiener's canonical expansion is the long crosscorrelation time required. The long crosscorrelation time results from the fact that the gaussian signal is statistically defined only over an infinite time. A conceptual solution of this difficulty is the design of special signals as discussed in section 2.4.

At this point, it is of interest to inquire whether functional expansions are useful for stability studies. It seems that the answer is negative and all attempts so far, to derive stability criteria from functional expansions confirm this answer. The reason is that all criteria derived from functional representations of input-output data are of a mathematical rather than physical nature and more often than not are overrestrictive. This can be appreciated through the following considerations:

The functional expansion of an input-output functional relationship is a generalized power series expansion of the output in terms of the input. Consequently, the existence of the expansion is defined by mathematical requirements and not by the physics of the system involved. For example, consider a system which has an output

$$y(\tau)= \frac{1}{1+\tau} \qquad \tau > 0 \qquad (41)$$

This output is a well behaved function for all values of τ and in fact $y \to 0$ ∞ $\tau \to \infty$. Suppose now that $y(\tau)$ is written in the form of a power series as

$$y(\tau)= 1 - \tau + \tau^2 - \tau^3 +\ldots \qquad (42)$$

As it is well known, this series exists and is convergent only if $|\tau| < 1$. The implication of this simple example is that if the convergence of $y(\tau)$ were derived from its power series expansion (42), the range of t would be unnecessarily limited even though the actual function $y(\tau)$ is bounded and convergent for all values of τ. This is exactly the difficulty that renders functional expansions of non-linear systems impractical for stability studies of reactor or any other systems.

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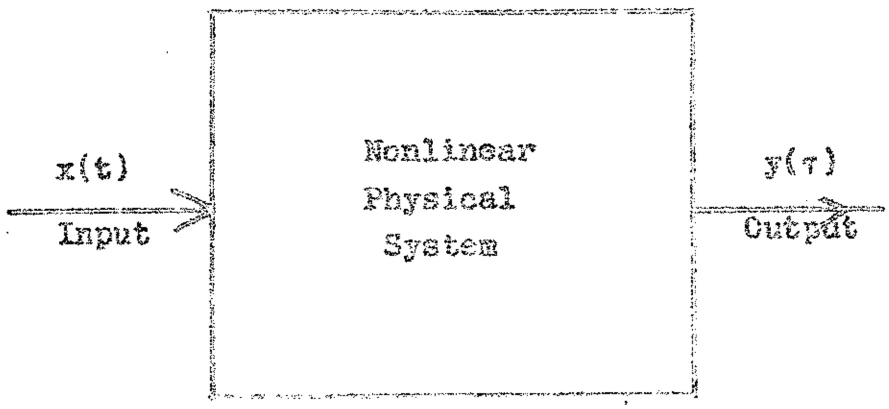

Fig. 1 Representation of a nonlinear system
$y(t) = F(x(t-\tau))\ ;\ \tau < t$

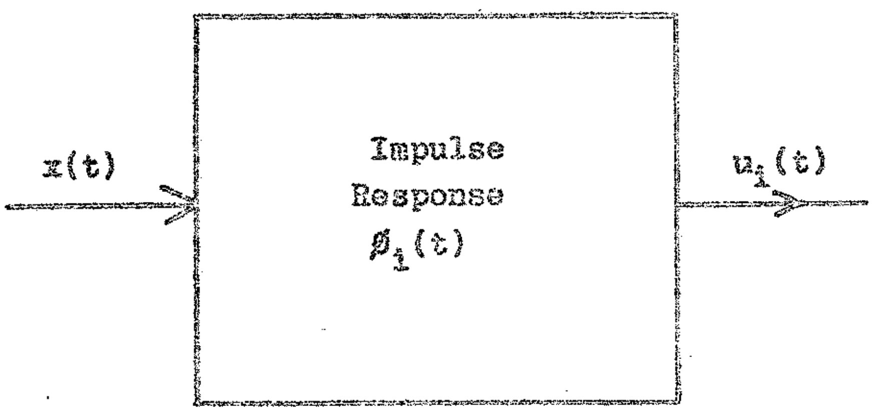

Fig. 2 Physical realization of coefficients
$u_i(t)$

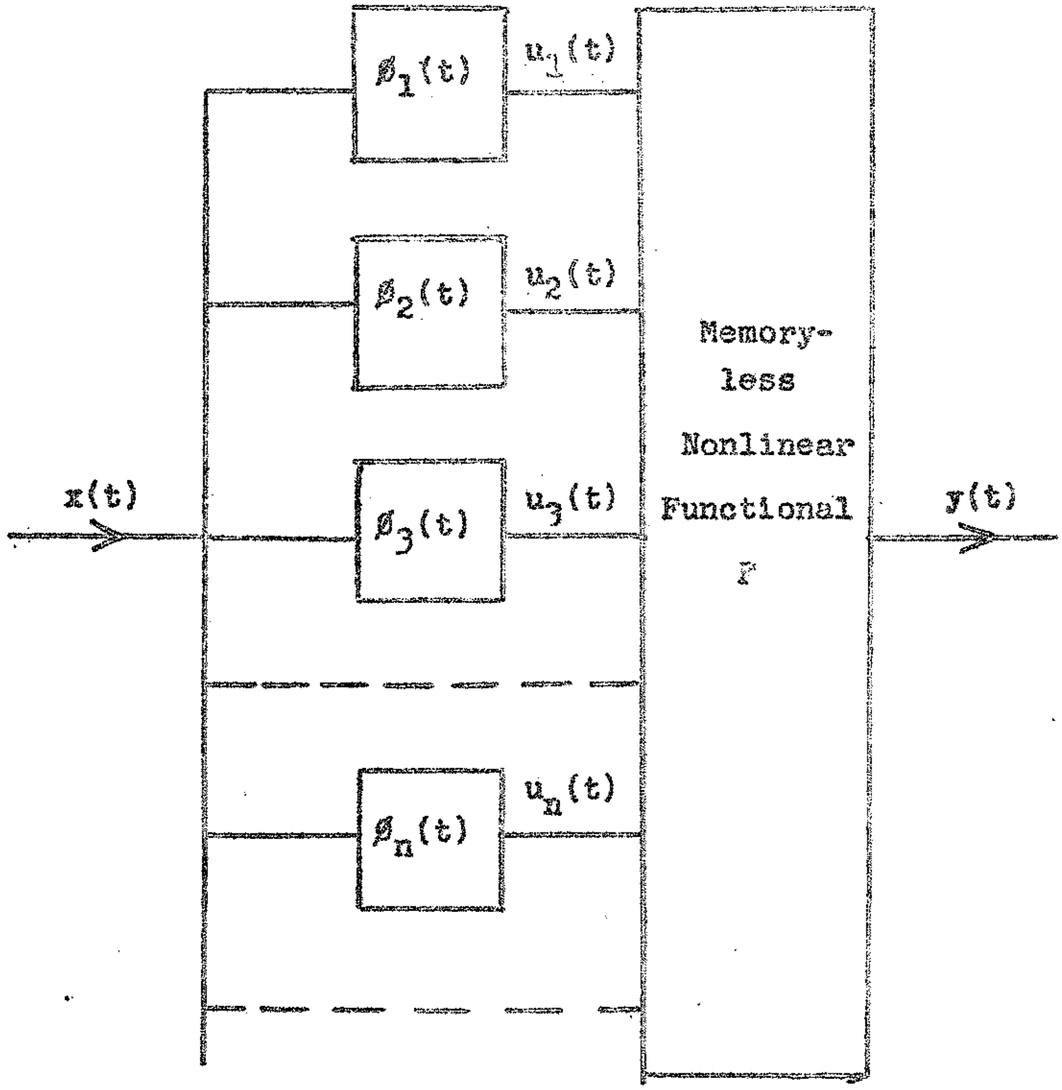

Fig. 3 Representation of a nonlinear system in terms of linear systems with memory and a nonlinear system without memory

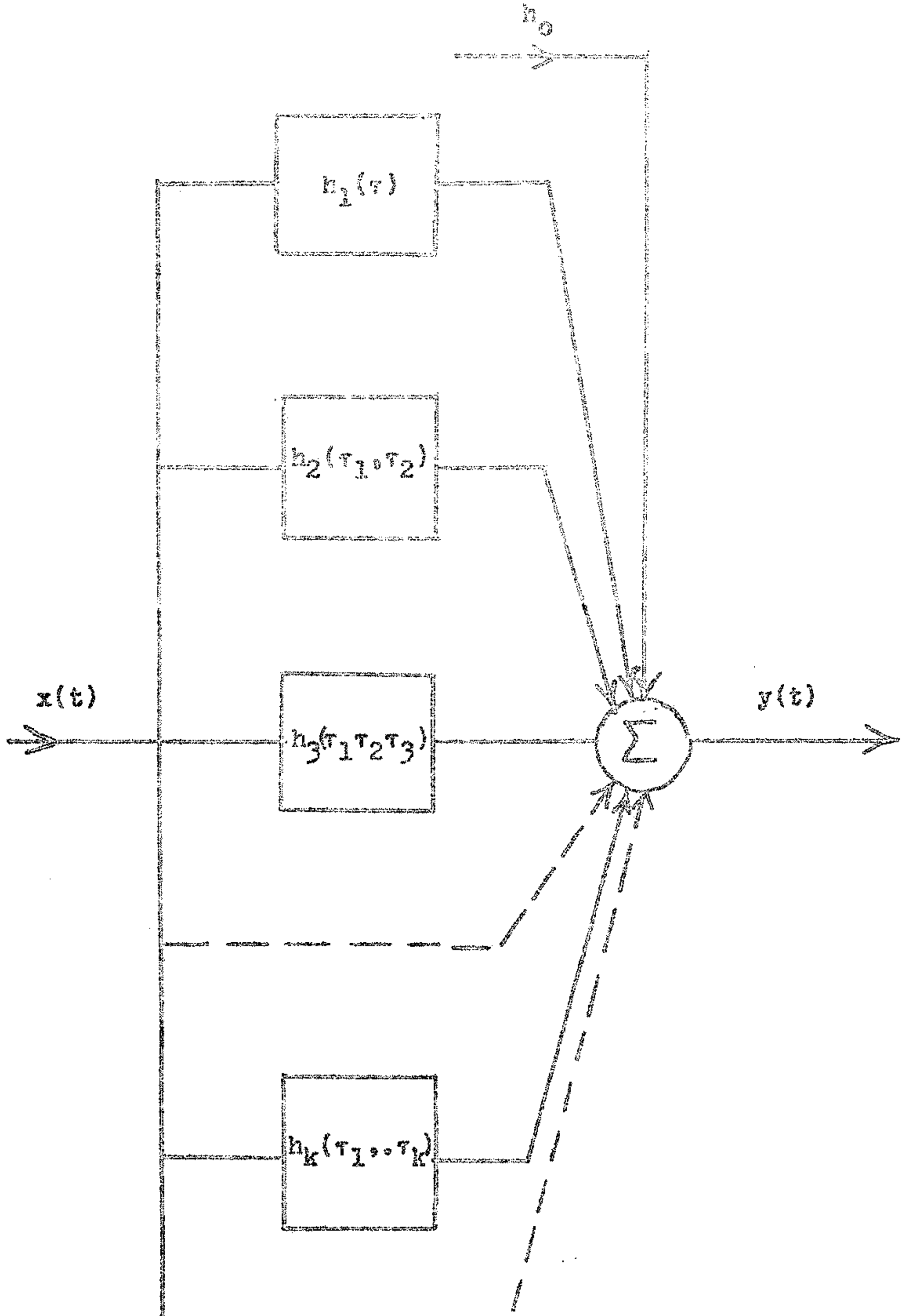

Fig. 4 Representation of a nonlinear system in terms of high order kernels.

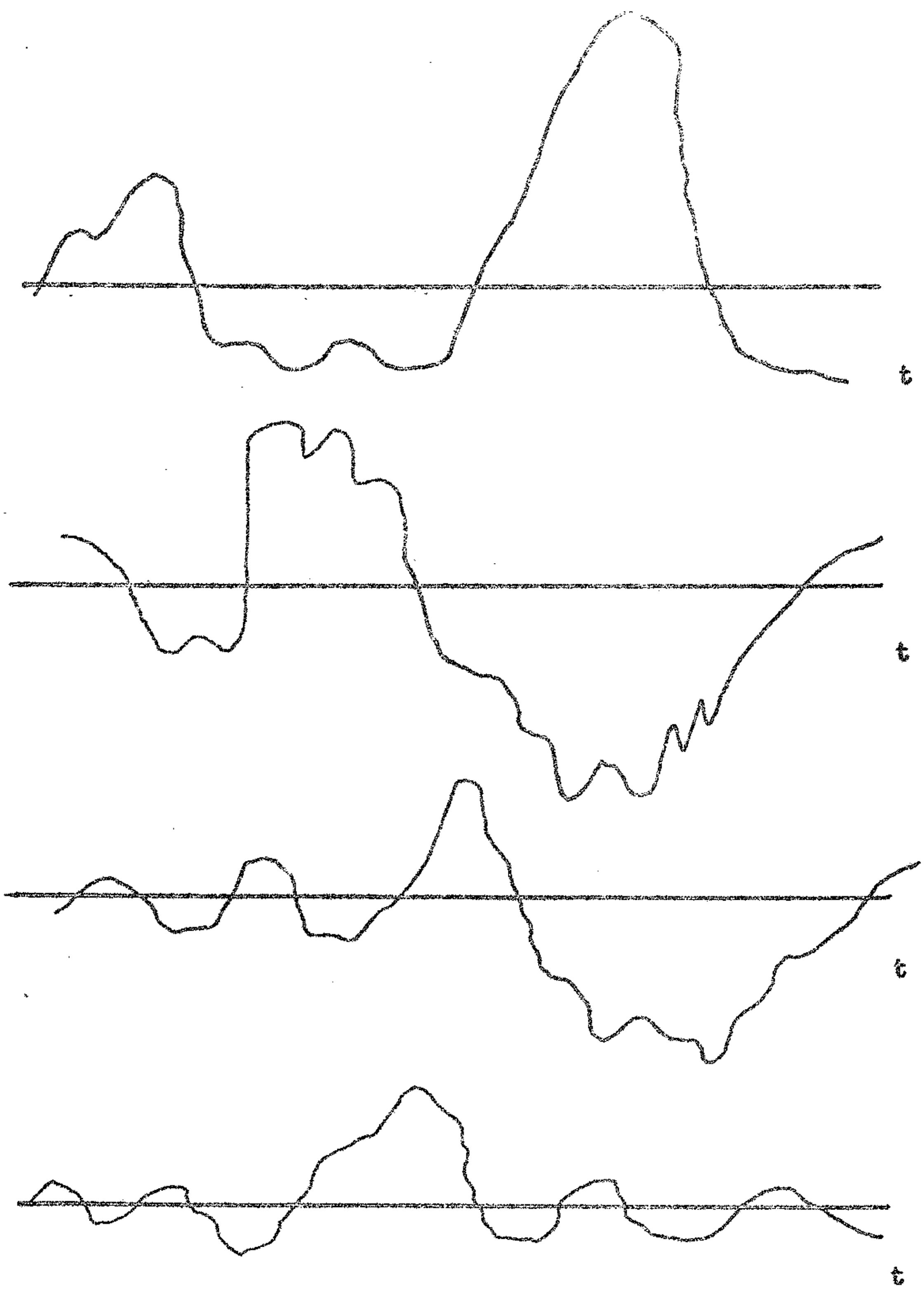

Fig. 5 Ensemble of Guassianly distributed white noise signals

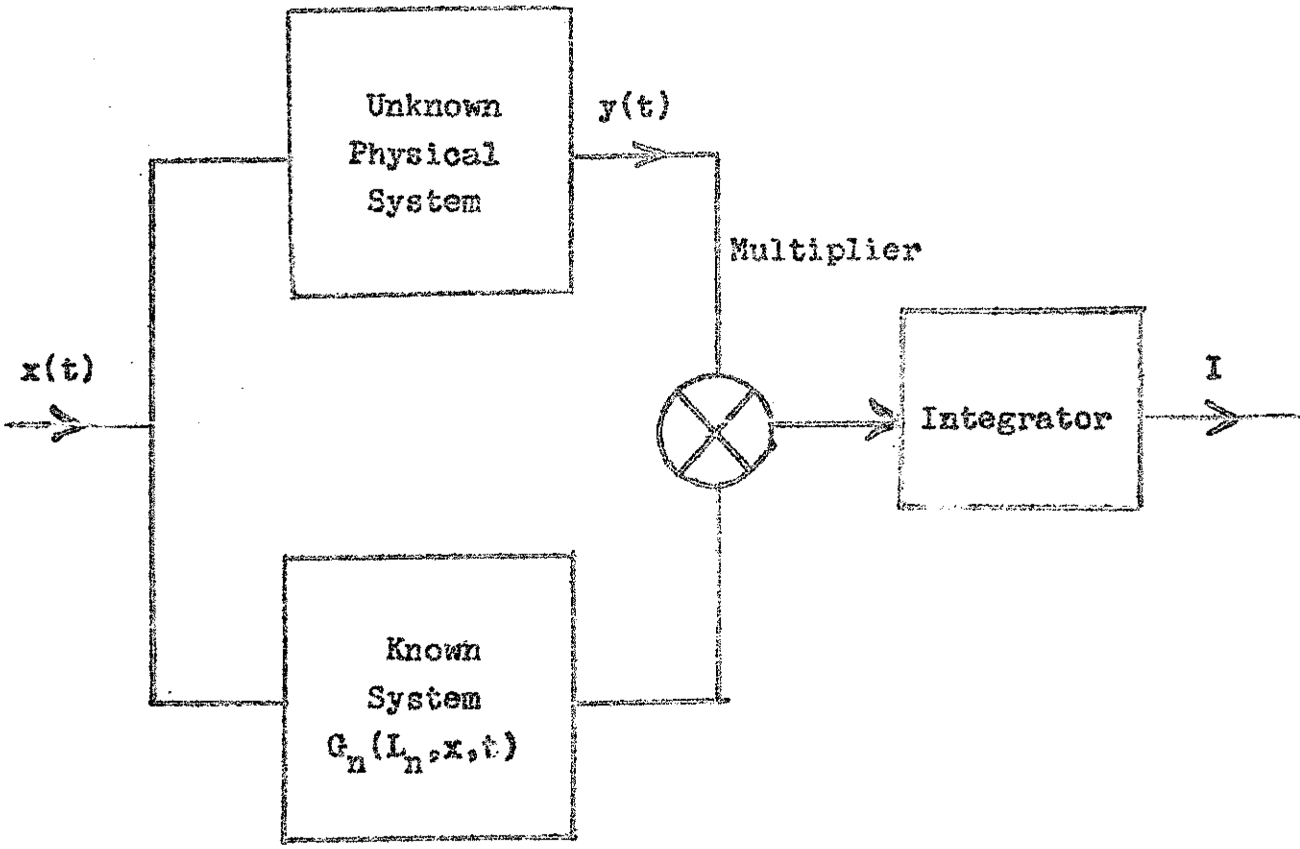

Fig. 6 Experimental set up for the measurement of the Wiener kernels

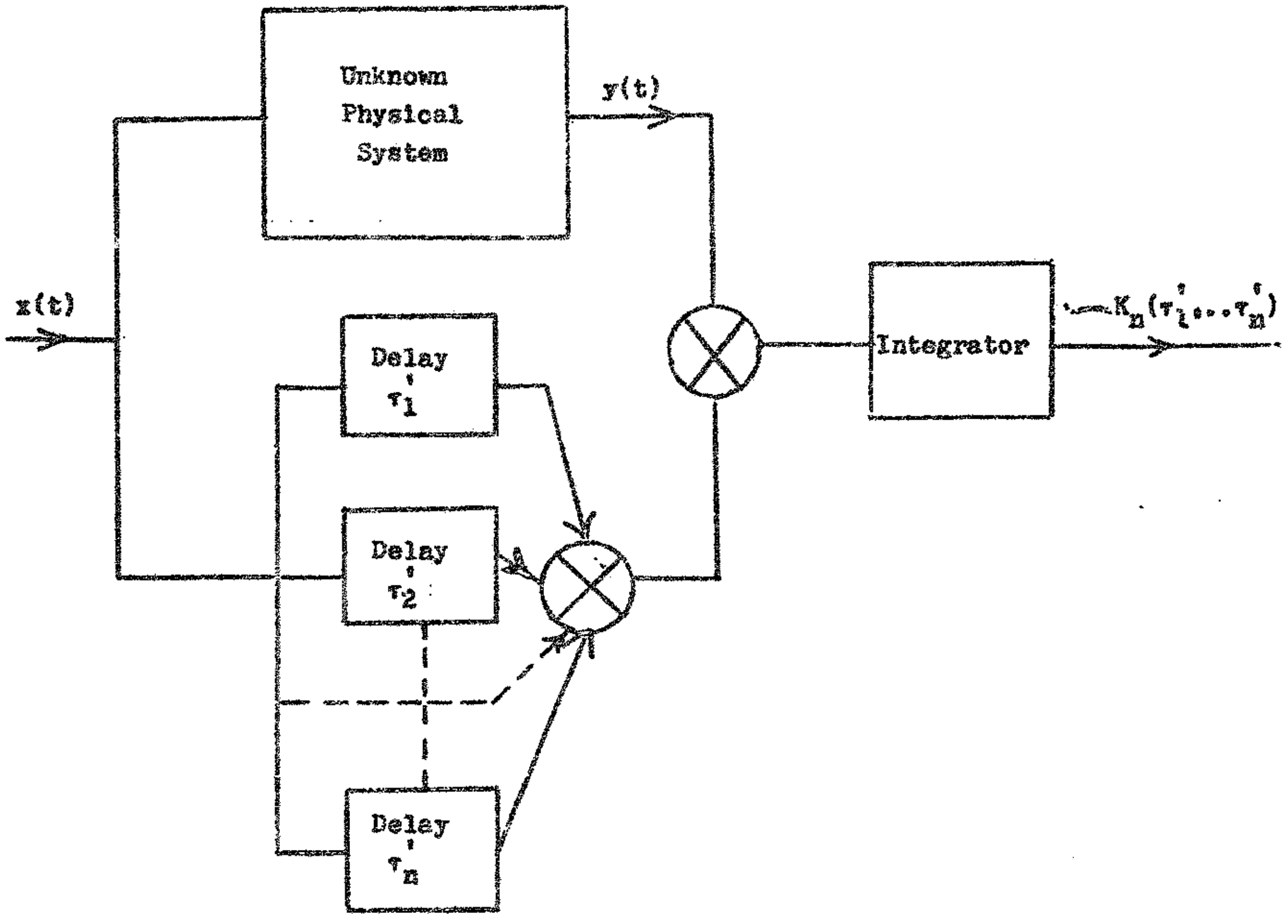

Fig. 7 Experimental set up for the measurement of the nth kernel of a nonlinear system

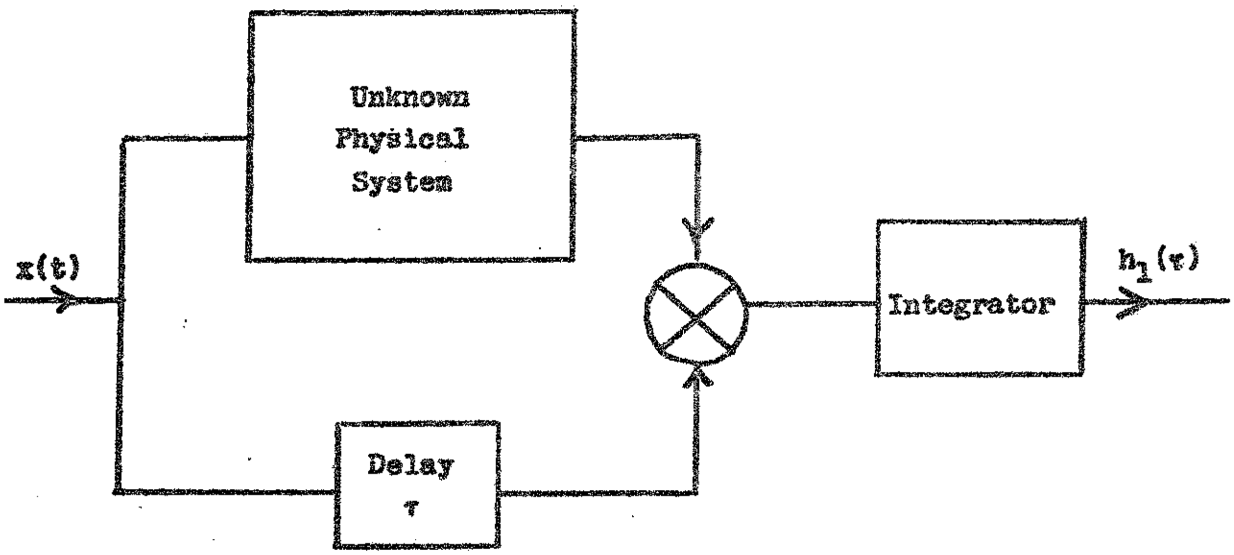

Fig. 8 Experimental set up for the measurement of the first Wiener kernel

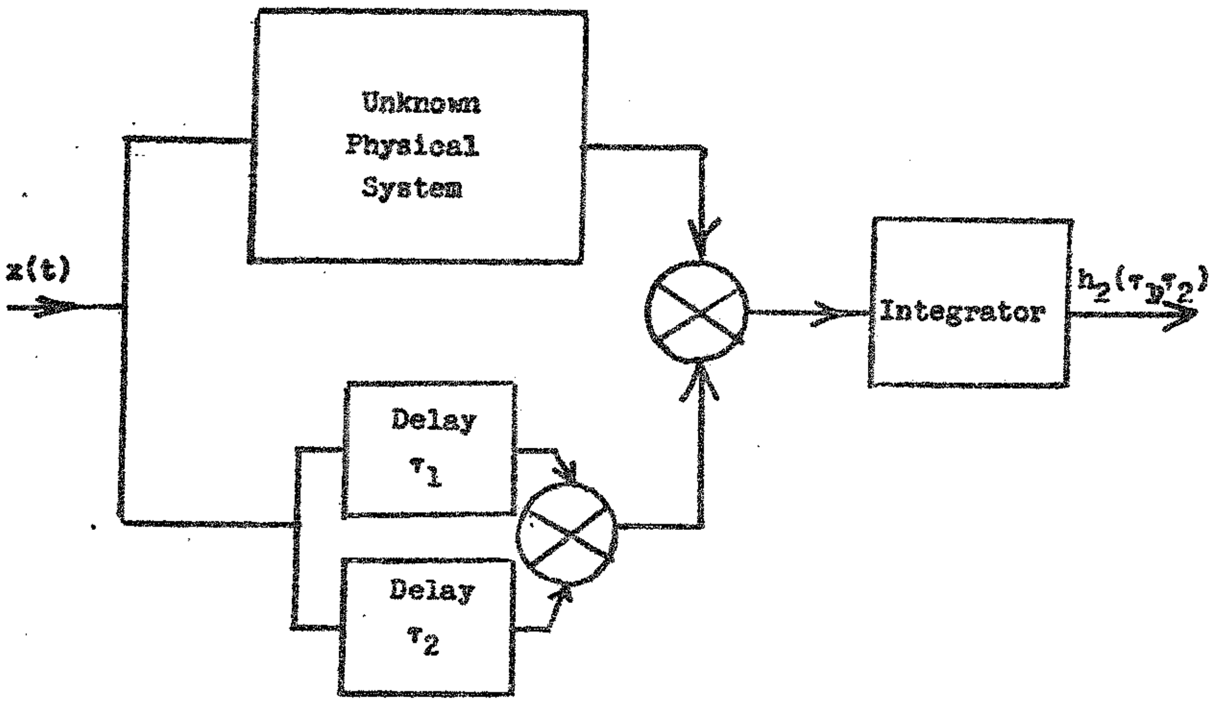

Fig. 9 Experimental set up for the measurement of the second Wiener kernel