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Section IX

Stability of Nonlinear Systems.

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by

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1. INTRODUCTION

The question of stability of nonlinear systems, from the analytical standpoint, is very involved. In contrast to linear systems, there is no unique necessary and sufficient criterion of stability for nonlinear systems and each class of nonlinear problems must be examined separately.

The purpose of these notes is to present some of the analytical techniques that are particularly applicable to systems whose dynamic behaviour is described by a set of ordinary nonlinear differential equations. Such techniques are of interest to nuclear engineers because many problems of nuclear reactor dynamics can be represented by nonlinear ordinary differential equations.

The following section presents the theory of Liapunov's direct method for stability (1). This theory has proved very fruitful particularly in the USSR where it is considered as the most general method for the investigation of problems of stability of nonlinear systems. The third section describes Lur/e's technique (2) for the implementation of Liapunov's theory in the case of a very general class of nonlinear systems which are similar to nuclear reactors. The fourth section discusses several applications of Liapunov's method in nuclear reactor problems and shows that Welton's criterion of stability (3) can be considered as a special case of Liapunov's direct method. The last section discusses the question of Lagrange stability of nonlinear systems.

2. STABILITY BY LIAPUNOV'S DIRECT METHOD

2.1 The Problem

A dynamical system, be it electrical, mechanical, nuclear etc. or any combination of these, can be described by a certain number of n parameters $x_1, x_2, \ldots x_n$. These parameters may be visualized as the coordinates of a point M in an n-dimensional space or equivalently, as the components of a vector x in the same space. Thus, each point of the n-dimensional space represents a state of the physical system.

Without loss of generality it may be assumed that the point $x = 0$ is an equilibrium or steady state. One of the fundamental questions of the theory of control is the type of stability of the equilibrium state. Specifically, if at time $t = 0$ the system is perturbed from its equilibrium state $(x(t=0)\neq 0)$ the question arises as to whether for $t\to\infty$ the parameters of the system resume their equilibrium values (the system is asymptotically stable) are bounded (the system is stable) or diverge (the system is unstable).

This question can be elegantly answered by means of Liapunov's direct method, if the dynamics of the system are adequately represented by a set of ordinary nonlinear differential equations. A large number of problems of engineering interest fall into this category. The literature and particularly the Russian literature are abundant in problems whose stability has been investigated by means of Liapunov's method (4 - 10).

To make ideas specific, assume that the dynamic behaviour of an autonomous* system is represented by the set of n ordinary differential equations

$$\frac{dx_k}{dt} = \dot{x}_k = X_k(x_1,\ldots x_n) \qquad k = 1,2,\ldots n \qquad (1)$$

or by the equivalent vector equation

$$\dot{x} = X(x) \qquad (2)$$

where X_k is a nonlinear function of $x_1,\ldots x_n$. In addition, suppose that $x = 0$ is an equilibrium state i.e. $X(0) = 0$. Geometrically speaking the meaning of eq. (2) is that the state of the system is represented by the point $M(x_1,\ldots x_n)$ and the velocity of the point M is $\dot{x}(X_1,\ldots X_n)$. In Liapunov's terminology eqs. (1) or (2) are called the equations of the disturbed motion.

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* Similar procedures have been developed for non-autonomous systems (5).

Let the quantities $x_{10}, \ldots x_{no}$ denote the initial values of the parameters $x_1, \ldots x_n$ at time $t = 0$. Corresponding to each set of initial values, assume that there exists a unique solution of eq. (2).

$$x_k = x_k(\tau, x_{10}, \ldots x_{no}) \qquad k = 1,2,\ldots n \qquad (3)$$

The solutions given by eq. (3) describe the disturbed motion of the system.

As already mentioned, an important question that must be answered with regard to the disturbed motion is its behaviour at $t \to \infty$. If it were possible to actually solve eq. (2), then all the information about the disturbed motion would be known. In general, however, this is a difficult task. Therefore, it is necessary to resort to qualitative techniques which survey the entire family of the disturbed motions and, without integration of eq. (2), permit to ascertain whether these motions tend to the quilibrium state or not as $t \to \infty$, regardless of the initial values x_{ko}. Such a technique is Liapunov's direct method which is subsequently described.

2.2 Liapunov's Direct Method.

Liapunov's direct method of stability is based on the existence of a positive definite scalar function $V(x)$ with the following properties.

a. $V(x)$ is continuous together with its first partial derivatives in a certain open region Ω about the origin $x = 0$.

b. $V(0) = 0$, $V(\infty) \to \infty$.

c. Outside the origin (and always in Ω) $V(x)$ is positive.

In other words $V(x)$ is non-negative and vanishes only at the origin. The origin is an isolated minimum of $V(x)$. If in addition, $\dot{V} \leq 0$ in Ω, then $V(x)$ is called a Liapunov function.

Thus, Liapunov's main stability theorems may be summarized as follows:

I. Stability Theorem: If there exists in some neighbourhood Ω of the origin a Liapunov function V(x), then the origin is stable.

II. Asymptotic Stability Theorem. If in addition to the requirements of theorem I, $-\dot{V}$ is likewise positive definite then the stability is asymptotic.

III. Instability Theorem. Let V(x) with V(0) = 0 have continuous first partials in Ω. Let $\dot{V}$ be positive definite and V(x) be able to assume positive values arbitrarily near the origin. Then the origin is unstable.

There are other variations and generalizations of Liapunov's theorems. For these, however, as well as the proof and geometric interpretation of theorems I-III the reader is referred to the literature (11-12). For the purposes of these notes suffice to note that the existence of a Liapunov function guarantees the stability of the origin or what has been assumed as the equilibrium state of the system described by eq. (2).

It must be emphasized that stability, asymptotic stability etc., of an equilibrium state of a physical system, do not necessarily imply the existence of Liapunov functions. However, from a practical standpoint this is not important. A particular Liapunov function yields certain sufficient conditions for stability which is what one wants in practice. Of course, it is also desirable to select the Liapunov function which results in the least restrictive conditions possible.

Having available a technique to investigate the stability of disturbed motions, the next question is "given a specific system of equations of the disturbed motion, how does one construct a Liapunov function ?" This problem has been dealt with by many authors. In particular, Lur'e has developed a general procedure which is the subject of discussion of the next section.

3. THE LUR'E THEOREM ON THE STABILITY OF CONTROL SYSTEMS.

The purpose of this section is to summarize the Lur'e theorem for the construction of Liapunov functions for a large class of nonlinear equations which are representative of many practical control problems.

Specifically, consider control systems whose equations of disturbed motion are of the form:

$$\dot{x}_k = \sum_{\alpha=1}^{m} b_{k\alpha} x_\alpha + n_k \mu \qquad k = 1,2\ldots m \qquad (4)$$

where $x_1, \ldots x_m$ are the system parameters

μ is the coordinate of the regulating organ

$b_{k\alpha}, n_k$ are constant coefficients

The coordinate of the regulating organ obeys the equation

$$V^2\ddot{\mu} + \dot{\mu} + S\mu = f(\sigma)$$

$$\sigma = \sum_{\alpha=1}^{m} p_\alpha x_\alpha - r'\mu \qquad (5)$$

where V^2, S, p_α, r' are constants and $f(\sigma)$ belongs to either of the following two classes:

Class (A) $\quad f(\sigma) = 0 \qquad |\sigma| \leq \sigma^*$

$\qquad \sigma f(\sigma) > 0 \qquad |\sigma| > \sigma^*$

Class (A^1) $\quad \sigma^* = 0 \qquad \left.\frac{df}{d\sigma}\right|_{\sigma=0} \geq h > 0$

$\qquad \sigma\Phi(\sigma) > 0 \qquad \Phi(\sigma) = f(\sigma) - h\sigma$

In summary, the collective equations of the disturbed motion of the system under consideration are:

$$\dot{x} = Bx + N\mu$$

$$V^2\ddot{\mu} + \dot{\mu} + S\mu = f(\sigma) \qquad (6)$$

$$\sigma = \sum_{\alpha=1}^{m} p_\alpha x_\alpha - r'\mu$$

where B is the mxm-matrix of the coefficients $b_{k\alpha}$

N is the m-element column matrix of the coefficients n_k

Next consider the equilibrium states of (6) given by the solution of the system of algebraic equations:

$$
\begin{aligned}
&Bx + N\mu = 0 \\
&S\mu = f(\sigma) \qquad (7) \\
&\sigma = \sum_{\alpha=1}^{m} p_\alpha x_\alpha - r'\mu
\end{aligned}
$$

For the purposes of this discussion it is algebraically expedient and by no means restrictive to assume that $f(\sigma)$ is of class (A^1) and that the equilibrium state is

$$x_1 = x_2 = \dots = \mu = \sigma = 0 \qquad (8)$$

The construction of a Liapunov function for the study of the stability of the equilibrium state is greatly facilitated if the system of equations (6) is transformed into a canonical form. The transformation is achieved by means of appropriate linear combinations of the parameters x_k. To see this, consider the following two cases:

a. All the characteristic roots ρ_k of the matrix B are distinct and have the property $\mathrm{Re}\rho_k < 0$. In other words, assume that the controlled system, with the regulating organ disconnected, has all its poles in the left half complex plane, namely that it is inherently stable. Thus, admit with Lur'e that equations (6) can be readily transformed into the canonical form:

$$
\begin{aligned}
&\dot{x}_k = \rho_k x_k + f(\sigma) \qquad k = 1,2\dots n;\ n = m + 2 \\
&\sigma = \sum_{k=1}^{n} \gamma_k x_k \qquad (9) \\
&\dot{\sigma} = \sum_{k=1}^{n} \beta_k x_k - rf(\sigma)
\end{aligned}
$$

where x_k is used again to denote the transformed parameters
β_k, γ_k, r are constants derived from the original coefficients
ρ_{m+1}, ρ_{m+2} are the roots of the equation

$$V^2\rho^2 + \rho + S = 0 \tag{10}$$

Notice that $Re\rho_{m+1}, Re\rho_{m+2} < 0$ ($S>0$). Lur'e gives explicit formulas for the transformation matrix and the coefficients β_k, γ_k, r in terms of the original coefficients and the characteristic roots (4).

A simple Liapunov function can now be constructed for the canonical system (9). To prove this assertion assume, that among the n characteristic roots ρ_k, there are s real $(\rho_1, \ldots \rho_s)$ and $(n-s)/2$ complex conjugate pairs $(\rho_{s+1}, \ldots \rho_n)$*. Consider the function

$$V(x) = F + \Phi + \int_0^{\sigma} f(\sigma)\, d\sigma \tag{11}$$

where F and Φ are quadratic forms. The quadratic form F is defined as:

$$F(a_1x_1, \ldots a_nx_n) = -\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{a_ka_i}{\rho_k+\rho_i} x_kx_i \tag{12}$$

where $a_1, \ldots a_s$ are arbitrary real numbers and $a_{s+1}, \ldots a_n$ are arbitrary complex conjugate sets of pirs. Notice that F is a positive definite quadratic form and that it vanishes only at the origin because

$$-\frac{1}{\rho_k+\rho_i} = \int_0^{\infty} e^{(\rho_k+\rho_i)\tau} d\tau \tag{13}$$

$$F = \sum_{i=1}^{n} \sum_{k=1}^{n} \left[a_ka_i \int_0^{\infty} e^{(\rho_k+\rho_i)\tau}\, d\tau\right] x_kx_i =$$

$$= \int_0^{\infty} \left[\sum_{k=1}^{n} a_kx_k e^{\rho_k\tau}\right]^2 d\tau$$

The quadratic form Φ is defined as:

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* It is evident that the corresponding canonical parameters x_k must be similar in nature.

$$\Phi(x_1,\ldots x_n) = \tfrac{1}{2}\sum_{k=1}^{s} A_k x_k^2 + C_1 x_{s+1}\, x_{s+2} + \ldots + C_{n-1-s} x_{n-1} x_n \tag{14}$$

where $A_1,\ldots A_s, C_1, C_3 \ldots C_{n-1-s}$ are positive real numbers. Notice again that Φ is a positive definite quadratic form and that it vanishes only at the origin.

In view of the fact that $f(\sigma)$ is of class (A^1) it is evident that V (eq.(11)) is a positive definite function that vanishes only at the origin and grows indefinitely as x goes to infinity. V will be a Liapunov function if in addition it is established that $\dot{V} \leq 0$. According to equations (9):

$$\dot{V} = \sum_{k=1}^{s} A_k x_k \left[\rho_k x_k + f(\sigma)\right] + C_1 x_{s+1}\left[\rho_{s+2} x_{s+2} + f(\sigma)\right] +$$

$$+ C_1 x_{s+2}\left[\rho_{s+1} x_{s+1} + f(\sigma)\right] + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{a_k a_i}{\rho_k + \rho_i}\Big[x_k\,[\rho_i x_i + f(\sigma)] +$$

$$+ x_i\,[\rho_k x_k + f(\sigma)]\Big] + f(\sigma)\left[\sum_{k=1}^{n} \beta_k x_k - r f(\sigma)\right] = -\Big[\sum_{k=1}^{n} a_k x_k +$$

$$+ \sqrt{r} f(\sigma)\Big]^2 + \sum_{k=1}^{s} \rho_k A_k x_k^2 + C_1(\rho_{s+1} + \rho_{s+2}) x_{s+1} x_{s+2} + \ldots C_{n-1-s}(\rho_{n-1} +$$

$$+ \rho_n) x_{n-1} x_n + f(\sigma) \sum_{k=1}^{s}\left[A_k + \beta_k + 2\sqrt{r}a_k - 2a_k \sum_{i=1}^{n} \frac{a_i}{\rho_k + \rho_i}\right] x_k +$$

$$+ f(\sigma) \sum_{\alpha=1}^{n-s}\left[C_\alpha + \beta_{s+\alpha} + 2\sqrt{r}a_{s+\alpha} - 2a_{s+\alpha} \sum_{i=1}^{n} \frac{a_i}{\rho_{s+\alpha} + \rho_i}\right] x_{s+\alpha} \tag{15}$$

where for convenience in writing the sums in the last bracket it is assumed that

$$C_1 = C_2,\ C_3 = C_4 \ldots \qquad\qquad C_{n-1-s} = C_{n-s}$$

and the quantity $2\sqrt{rF(\sigma)}\sum_{k=1}^{n} a_k x_k$ has been added and subtracted from the equation.

Notice that if

$$A_k + \beta_k + 2\sqrt{r}a_k - 2a_k \sum_{i=1}^{n} \frac{a_i}{\rho_k+\rho_i} = 0 \qquad k=1,2\ldots s$$

$$C_\alpha + \beta_{s+\alpha} + 2\sqrt{r}a_{s+\alpha} - 2a_{s+\alpha} \sum_{i=1}^{n} \frac{a_i}{\rho_{s+\alpha}+\rho_i} = 0 \qquad \alpha = 1,2\ldots(n-s) \tag{16}$$

then $\dot{V}$ is negative definite since $\rho_k<0$ $(k=1,2\ldots s)$ and $\operatorname{Re}\rho_{s+\alpha}<0$ $(\alpha=1,2,\ldots n-s)$. Consequently, V is a Liapunov function and the equilibrium state is asymptotically stable.

The meaning of conditions (16), first derived by Lur'e may be stated in terms of the following theorem: If eqs. (16) in which $\rho_1,\ldots\rho_s$ and $\beta_1,\ldots\beta_s$ are real and $\rho_{s+1},\ldots\rho_n$ and $\beta_{s+1},\ldots\beta_n$ are complex conjugate pairs, admit real roots $a_1,\ldots a_s$ and complex conjugate pairs of roots $a_{s+1},\ldots a_n$ for arbitrary positive numbers A_k, C_α, then the equilibrium state of the system described by (6) is absolutely and asymptotically stable. For this to be true, the constant coefficients of the system must satisfy specific algebraic relationships.

The derived conditions can be modified in several ways. For further details, the reader is referred to (4).

It is important to note here that in all cases, Lur'e's theorem requires asymptotic stability in the small (for small perturbations when the system (6) can be linearized) in order to guarantee absolute stability. In addition, in several specific problems that Letov (4) has analyzed by Lur'e's method, it turns out that the necessary and sufficient conditions for the satisfaction of eqs. (16) are the same as the necessary and sufficient conditions for the linearized system (6) to be absolutely stable. Unfortunately, this result has not been generalized but its importance is of such practical value that it is worth further consideration.

Now consider the second case.

b. Some of the characteristic roots of B have positive real parts ($Re\rho_k > 0$) i.e. the controlled system is inherently unstable. To proceed with the construction of a Liapunov function admit with Letov (4) that the augmented matrix $\bar{B}$ with elements

$$\bar{b}_{k\alpha} = b_{k\alpha} + \frac{n_k \rho_\alpha}{r} \tag{17}$$

has distinct characteristic roots r_s and such that $Re r_s < 0$. Thus, transform the equations of the disturbed motion into the canonical form

$$\begin{aligned} \dot{x}_s &= r_s x_s + \sigma \\ \dot{\sigma} &= \sum_{k=1}^{m} \bar{\beta}_k x_k - \bar{\rho}\sigma - f(\sigma) \end{aligned} \tag{18}$$

The existence of the transformation, the transformation matrix and explicit formulas for the coefficients $\bar{\beta}_k$ and $\bar{\rho}$ are given by Letov (4)

Following arguments and assumptions similar to those used in the case of inherently stable systems, notice that the positive definite function

$$V = -\sum_{k=1}^{m}\sum_{i=1}^{m} \frac{a_k a_i}{r_k + r_i} x_k x_i + \Phi(x_1, \ldots x_m) + \tfrac{1}{2} d^2 \sigma^2 \tag{19}$$

is a Liapunov function if the following conditions are satisfied:

$$A_k + d^2 \bar{\beta}_k + 2a_k - 2a_k \sum_{i=1}^{m} \frac{a_i}{r_k + r_i} = 0 \qquad k=1,2,\ldots s$$

$$C_\alpha + d^2 \bar{\beta}_{s+\alpha} + 2a_{s+\alpha} - 2a_{s+\alpha} \sum_{i=1}^{m} \frac{a_i}{r_{s+\alpha} + r_i} = 0 \qquad \alpha=1,2,\ldots m-s \tag{20}$$

$$d^2\bar{\rho} - 1 > 0$$

Indeed, under these conditions

$$\dot{V} = -\left[\sum_{k=1}^{m} a_k x_k + \sigma\right]^2 - \left[d^2\bar{\rho}-1\right]\sigma^2 - d^2\ \sigma f(\sigma) + \sum_{k=1}^{s} r_k A_k x_k^{\,2} +$$

$$+ C_1(r_{s+1} + r_{s+2})x_{s+1}x_{s+2} + \ldots + C_{m-s-1}(r_{m-1}+r_m)x_{m-1}x_m < 0 \qquad (21)$$

The meaning of conditions (20) is that the equilibrium state $x=\sigma=0$ is absolutely and asymptotically stable if eqs. (20) admit real roots $a_1, \ldots a_s$ and complex conjugate pairs of roots $a_{s+1}, \ldots a_m$ for A_k, C_α positive, $\bar{\beta}_1, \ldots \bar{\beta}_s$ real and $\bar{\beta}_{s+1}, \ldots \bar{\beta}_m$ complex conjugate. This is Letov's generalization of Lur'e's theorem. Note again that stability in the small is required to guarantee absolute stability.

This completes the brief discussion of Lur'e's theorem and Letov's generalization. Many more details can be found in (4), where in addition systems whose canonical equations are of the form

$$\begin{aligned}
\dot{x}_k &= \rho_k x_k + u_1^{(k)}\ f_1(\sigma_1) + u_2^{(k)}\ f_2(\sigma_2) \qquad k=1,2,..n \\
\dot{\sigma}_1 &= \sum_{\alpha=1}^{n} \beta_{1\alpha} x_\alpha - r_{11} f_1(\sigma_1) - r_{12} f_2(\sigma_2) \qquad (22) \\
\dot{\sigma}_2 &= \sum_{\alpha=1}^{n} \beta_{2\alpha} x_\alpha - r_{21} f_1(\sigma_1) - r_{22} f_2(\sigma_2)
\end{aligned}$$

are analyzed by the same procedure.

In the next section it is shown that a variety of problems on nuclear reactor stability can be reduced to canonical forms similar to those of eqs. (9) or (18) and that they can be readily analyzed by means of Liapunov's direct method.

4. APPLICATIONS OF LIAPUNOV'S DIRECT METHOD TO NUCLEAR REACTOR DYNAMICS

4.1 General Remarks

Consider a nuclear reactor system whose dynamic behaviour is adequately represented by the following set of ordinary differential equations

$$\frac{d\Phi}{dt} = - f(x_1, x_2 \ldots x_k, \Phi)\Phi - \frac{\beta}{\Lambda}\Phi + \sum_i^m \lambda_i C_i$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda}\Phi - \lambda_i C_i \qquad i = 1,2,\ldots m \tag{23}$$

$$\frac{dx_i}{dt} = P_i(x_1, \ldots x_k, \Phi) \qquad i=1,2,\ldots k$$

The question of stability of the equilibrium states of this system can be readily investigated by means of Liapunov's direct method, for different types of functions f and P_i. To this effect it is often expedient to reduce the complexity of eqs. (23) by neglecting the delayed neutron precursors. This omission is justified because it has been shown that the delayed neutron precursors tend always to increase the stability of the system (13). In addition, Popov (6) has shown that if the system without delayed neutrons

$$\frac{d\Phi}{dt} = - f(x_1, x_2, \ldots x_k, \Phi)\Phi \tag{24}$$

$$\frac{dx_i}{dt} = P_i(x_i, \ldots x_k, \Phi)$$

admits a Liapunov function

$$V_1 = V_a(\Phi) + V_b(x_1, x_2, \ldots x_k) \tag{25}$$

where $V_a(\Phi)$ is an increasing function of Φ, then

$$V_2 = V_a(\Phi) + V_b(x_1, \ldots x_k) + \sum_{i=1}^{m} \frac{\beta_i}{\lambda_i \Lambda} V_a\left(\frac{\lambda_i \Lambda}{\beta_i} C_i\right) \tag{26}$$

is a Liapunov function for the system of eqs. (23) with the delayed neutron precursors.

Popov's theorem is useful in the context of these notes because in many problems of reactor dynamics, omission of the delayed neutron precursors allows the reduction of eqs. (24) to one of the canonical forms (9) or (18) and, consequently, the derivation of the Liapunov function (25) by means either of Lur'e's theorem or Letov's generalization of Lur'e's theorem.

Of course, it must also be emphasized that omission of the delayed neutron precursors leads to more restrictive conditions for stability than if Liapunov's direct method were implemented for the system including the delayed neutron precursors (14). This goes to prove once more that one never gets something for nothing!

Some of these points are best illustrated by the following examples.

4.2 Stability of Heterogeneous Reactors.

Consider a n region reactor. Each region j is characterized by an average temperature T_j and a coefficient of reactivity (over the neutron mean lifetime) α_j. The space independent model of the reactor is

$$\frac{d\Phi}{dt} = \left(\sum_j \alpha_j T_j\right)\Phi - \frac{\beta}{\Lambda}\Phi + \sum_i^m \lambda_i C_i$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda}\Phi - \lambda_i C_i \qquad (27)$$

$$\epsilon_j \frac{dT_j}{dt} = \eta_j \left(\frac{\Phi}{\Phi_o} - 1\right) - \sum_k x_{jk}(T_j - T_k)$$

where ϵ_j the heat capacity of the jth region

η_j the fraction of power developed in the jth region

x_{jk} the thermal conductivity between the jth and kth regions.

If the delayed neutrons are neglected and the change of variable

$$\Phi = \Phi_o e^{\sigma} \qquad (28)$$

is introduced, eqs. (27) reduce to

$$\dot{\sigma} = \sum a_j T_j$$

$$\epsilon_j \frac{dT_j}{dt} = -\sum x_{jk}(T_j - T_k) + \eta_j f(\sigma) \tag{29}$$

$$f(\sigma) = e^{\sigma} - 1; \; \sigma f(\sigma) > 0$$

It is evident that eqs. (29) are of the general form considered in section (3) and therefore necessary and sufficient conditions for the existence of a Liapunov function can be readily established by means of Lur'e's theorem.

Specific examples of this type have been treated in (15).

4.3 Welton's Criterion of Stability.

A general criterion of stability for nuclear reactors has been proposed by Welton (3). Specifically Welton considers a reactor whose dynamics are represented by the equations:

$$\frac{d\Phi}{dt} = -\frac{\rho(t)+\beta}{\Lambda}\Phi + \sum \lambda_i C_i$$
$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda}\Phi - \lambda_i C_i \tag{30}$$

where

$$\rho(t) = \int_{-\infty}^{t} g(t-\tau)\left[\Phi(\tau) - \Phi_o\right] d\tau$$
$$g(t) = 0 \qquad\qquad t < 0 \tag{31}$$

Welton showed that the reactor is absolutely stable if

$$G(\omega) \geq 0 \tag{32}$$

where $G(\omega) = \frac{1}{2\pi}\int_0^{\infty} g(\tau)\cos\omega\tau d\tau$ (33)

The meaning of inequality (32) is that the phase of the Fourier transform of the feedback kernel $g(t)$ be less than $\pm 90^\circ$. Note that this requirement is more restrictive than requiring that the linearized model of (30) be absolutely stable.

Welton's criterion can be derived by means of Liapunov's direct method (16). Indeed, consider eq. (30) without delayed neutrons.

$$\frac{1}{\Phi}\frac{d\Phi}{dt} = -\frac{1}{\Lambda}\rho(t) = -\frac{1}{\Lambda}\int_0^\infty G(\omega)\left[q(\omega,t)+q^*(\omega,t)\right]d\omega \qquad (34)$$

where

$$q(\omega,t) = \int_0^t e^{i\omega(t-\tau)}\left[\Phi(\tau)-\Phi_o\right]dt$$

$$q^*(\omega,t) = \text{conjgate of } q(\omega,t) \qquad (35)$$

If $G(\omega) \geq 0$, the function

$$V = \Phi-\Phi_o-\Phi_o \ln\frac{\Phi}{\Phi_o} + \frac{2}{\Lambda}\int_0^\infty G(\omega)|q(\omega,t)|^2 d\omega \qquad (36)$$

is positive definite and its derivative $\dot{V} = 0$. Since $(\Phi-\Phi_o-\Phi_o \ln \Phi/\Phi_o)$ is an increasing function of Φ, inclusion of the delayed neutron precursors by means of Popov's theorem renders $V < 0$ and therefore the reactor absolutely stable. Consequently, the sufficient condition for this to be true is $G(\omega) \geq 0$.

5. LAGRANGE STABILITY OF NONLINEAR SYSTEMS

5.1 General Remarks.

As it has already been emphasized Lur'e's implementation of Liapunov's direct method requires that the linearized version of the nonlinear system under consideration be unconditionally stable. In addition Letov has shown that in several specific problems the sufficient conditions for absolute stability coincide with those for linear stability.

The same comments are true for Welton's criterion in the case of nuclear reactor systems. In other words, the sufficient condition $G(\omega) \geq 0$ is equivalent to at least requiring that the linearized version of the reactor equations admit unconditionally stable solutions.

Is was also indicated that under these conditions the non-linear stability is asymptotic.

These observations imply the important experimental fact that nonlinear stability can be experimentally investigated by transfer function measurements.

In many applications, where certain parameters must be kept within very close tolerances, asymptotic stability is necessary. However, in many other applications conditions for asymptotic stability may needlessly restrict the system designer. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many aircraft, missiles, nuclear power plants etc. behave in this way and yet their performance is not considered undesirable. Such systems may be classified as Lagrange stable systems and the question arises "how does one go about analyzing Lagrange stability?" It turns out that Liapunov's direct method is also helpful in answering this question. This is discussed in the next section.

5.2 Liapunov's Direct Method for Lagrange Stable Systems.

Mathematically speaking, Lagrange stability may be defined by a simple extension of Liapunov's direct method. Specifically, consider a physical system whose disturbed motion is described by eq. (2). If a positive scalar function $V(x)$ of the type defined in section 2 can be found and if in addition

$$\dot{V} \leq - \epsilon < 0 \qquad (37)$$

for all x outside some closed and bounded region around the equilibrium state, then the disturbed motion of (2) is ultimately bounded and the system possesses Lagrange stability.

It is obvious that whether a Lagrange stable system is acceptable for a practical application depends on the size of the bounded region defined above and the particular application.

This extension of Liapunov's direct method to Lagrange stable systems has been successfully applied to a large class of nonlinear systems of the type considered in section 3. Explicit conditions for Lagrange stability are derived and it is shown that they are much less restrictive than those required for asymptotic stability (17).

For the purposes of these notes, the procedure will be illustrated by an example taken from the nuclear reactor field.

5.3 Lagrange Stability of a Nuclear Reactor with Two Temperature Coefficients of Reactivity.

Consider a two-region reactor with two temperature coefficients of reactivity. Assume that the reactor dynamics model is independent of spatial coordinates and neglect the delayed neutrons. Thus, the step response of the reactor is described by the equations

$$\frac{d\Phi}{dt} = \rho_1 \Phi \tag{38}$$

$$\epsilon_1 \frac{dT_1'}{dt} = \eta_1 (\Phi - \Phi_o) - h(T_1' - T_2') \tag{39}$$

$$\epsilon_2 \frac{dT_2'}{dt} = \eta_2 (\Phi - \Phi_o) + h(T_1' - T_2') - wT_2' \tag{40}$$

$$\rho_1 = \rho_o + r_1' T_1' + r_2' T_2' \tag{41}$$

where

- ϵ_i is heat capacity of ith region
- h is over-all heat transfer coefficient between regions (1) and (2)
- η_i is fractional power delivered to ith region ($\eta_1 + \eta_2 = 1$)
- r_i' is temperature coefficient of reactivity over neutron lifetime

ρ_o is step input over neutron lifetime
T_i' is average temperature increment of ith region
wT_2' is power removal
Φ is power
Φ_o is steady-state power before step ρ_o is applied.

A simple change of variable

$$T_i = T_1' + b_i T_2' \tag{42}$$

where

$$b_{1,2} = \frac{\frac{h}{\epsilon_1} - \frac{h}{\epsilon_2} - w \pm \sqrt{\left[\frac{h}{\epsilon_1} - \frac{h}{\epsilon_2} - w\right]^2 + 4\frac{h^2}{\epsilon_1 \epsilon_2}}}{2\frac{h}{\epsilon_2}} \tag{43}$$

reduces the system of eqs. (38-41) into the form

$$\frac{d\Phi}{dt} = \rho_1 \Phi \tag{44}$$

$$\frac{dT_1}{dt} = \alpha_1 \left[\Phi - \Phi_o\right] - g_1 T_1 \tag{45}$$

$$\frac{dT_2}{dt} = \alpha_2 \left[\Phi - \Phi_o\right] - g_2 T_2 \tag{46}$$

$$\rho_1 = \rho_o + r_1 T_1 + r_2 T_2 \tag{47}$$

with

$$g_i = \frac{h}{\epsilon_1} - b_i \frac{h}{\epsilon_2} \qquad \alpha_i = \frac{\eta_1}{\epsilon_1} + b_i \frac{\eta_2}{\epsilon_2}$$

$$r_1 = \frac{r_2' - r_1' b_2}{b_1 - b_2} \qquad r_2 = \frac{r_1' b_1 - r_2'}{b_1 - b_2}$$

The coefficients g_i are always positive. The coefficients α_i can also be assumed positive because, if α_i were not, a simple change of variable $T_i \rightarrow - T_i$ would result in a system with positive coefficients.

The system of eqs. (44-47) admits an equilibrium state

$$\Phi_\infty = \Phi_o - \frac{\rho_o g_1 g_2}{r_1\alpha_1 g_2 + r_2\alpha_2 g_1} \qquad \rho = 0$$

$$T_{1\infty} = \frac{\alpha_1(\Phi_\infty - \Phi_o)}{g_1} \qquad T_{2\infty} = \frac{\alpha_2(\Phi_\infty - \Phi_o)}{g_2} \quad (48)$$

The equilibrium state is physically meaningful if

$$r_1\alpha_1 g_2 + r_2\alpha_2 g_1 < 0$$

If the variables are measured in terms of their equilibrium values, eqs. (44-47) reduce to the normalized form

$$\frac{d\Phi}{dt} = \rho\Phi \quad (49)$$

$$\frac{dT_1}{dt} = g_1(\Phi - T_1) + g_1\frac{\Phi_o}{\Phi_\infty - \Phi_o}(\Phi - 1) \quad (50)$$

$$\frac{dT_2}{dt} = g_2(\Phi - T_2) + g_2\frac{\Phi_o}{\Phi_\infty - \Phi_o}(\Phi - 1) \quad (51)$$

$$\rho = \left[\frac{r_1\alpha_1}{g_1}(T_1 - 1) + \frac{r_2\alpha_2}{g_2}(T_2 - 1)\right](\Phi_\infty - \Phi_o) \quad (52)$$

For the purposes of the subsequent discussion assume Φ_o equal to zero. This is done for mathematical expediency and does not involve any loss of generality.

Finally, if the origin of coordinates of the (Φ, T_1, T_2) space is transferred to the equilibrium point (1,1,1) and T_i/g_i is replaced by T_i, eqs (49-52) reduce to the canonical form

$$\frac{dT_1}{dt} = -g_1 T_1 + f(\sigma) \quad (53)$$

$$\frac{dT_2}{dt} = -g_2 T_2 + f(\sigma) \quad (54)$$

$$\frac{d\sigma}{dt} = \Phi_\infty(r_1\alpha_1 T_1 + r_2\alpha_2 T_2) \quad (55)$$

$$f(\sigma) = e^{\sigma}-1 \; ; \; \sigma f(\sigma) > 0 \tag{56}$$

Notice that eqs. (53-56) are of the same type as eqs. (9). Consequently, sufficient conditions for absolute stability are (see eqs. (16) with $A_k = C_\alpha = 0$):

$$\Phi_\infty r_1 \alpha_1 + \frac{a_1^2}{g_1} + \frac{2a_1 a_2}{g_1+g_2} = 0 \tag{57}$$

$$\Phi_\infty r_2 \alpha_2 + \frac{\alpha_2^2}{g_2} + \frac{2a_1 a_2}{g_1+g_2} = 0 \tag{58}$$

The next question is under what conditions do eqs. (57-58) admit real solutions $\alpha_1 \alpha_2$? To answer this question multiply eq. (57) by g_1 and (58) by g_2 and add the results. Thus find

$$\Phi_\infty(r_1\alpha_1 g_1 + r_2\alpha_2 g_2) + (a_1+a_2)^2 = 0 \tag{59}$$

Eq. (59) implies that $r_1\alpha_1 g_1 + r_2\alpha_2 g_2 < 0$, since α_1, α_2 must be real. Similarly if eqs. (57-58) are divided by g_1, g_2 respectively and then added

$$\Phi_\infty(r_1\alpha_1 g_2 + r_2\alpha_2 g_1) / g_1 g_2 + \left(\frac{a_1}{g_1} + \frac{a_2}{g_2}\right)^2 = 0 \tag{60}$$

which also implies that $r_1\alpha_1 g_2 + r_2\alpha_2 g_2 < 0$. In summary then the system is absolutely and asymptotically stable if

$$r_1\alpha_1 g_1 + r_2\alpha_2 g_2 < 0 \;\; ; \;\; r_1\alpha_1 g_2 + r_2\alpha_2 g_1 < 0 \tag{61}$$

It can be readily shown that these are also the necessary and sufficient conditions for the linearized version of eqs. (49-52) to admit unconditionally stable solutions.

The same conclusion could have been derived through application of Welton's criterion to eqs. (49-52) with $\Phi_o = 0$. Indeed, the feedback kernel is

$$g(t) = -\left(r_1\alpha_1 e^{-g_1 t} + r_2\alpha_2 e^{-g_2 t}\right) \qquad t > 0 \qquad (62)$$

$$g(t) = 0 \qquad t > 0$$

The cosine transform of the feedback kernel is

$$G(\omega) = -\left[\frac{r_1\alpha_1 g_1}{\omega^2+g_1^2} + \frac{r_2\alpha_2 g_2}{\omega^2+g_2^2}\right] \qquad (63)$$

and it is positive only when conditions (61) are satisfied. Therefore, Welton's criterion results in the same conditions as Lur'e's method.

Now consider what happens if conditions (61) are not completely satisfied. Of course, the requirement

$$r_1\alpha_1 g_2 + r_2\alpha_2 g_1 = -R_2 < 0 \qquad R_2 > 0$$

is necessary for the existence of the equilibrium state. Assume that

$$r_1\alpha_1 g_1 + r_2\alpha_2 g_2 = R_1 > 0 \quad \text{and} \quad r_1\alpha_1 + r_2\alpha_2 = -R_o < 0 \qquad (64)$$

It can be easily shown that when conditions (64) are satisfied, the linearized version of the reactor equations leads to a conditionally stable system. Specifically, the reactor is linearly stable (Φ_o=0) when the step reactivity input is

$$\rho_o < \frac{(g_1 + g_2)R_2}{R_1} \qquad (65)$$

or the equilibrium power is

$$\Phi_\infty < \frac{g_1\ g_2\,(g_1 + g_2)}{R_1} \qquad (66)$$

It will be subsequently shown that the reactor is actually Lagrange stable for all values of the step reactivity ρ_o.

To this effect consider the system of eqs. (49-52) (Φ_o= 0) and introduce the change of variables

$$\sigma = \ln(\Phi+1) \qquad \Phi = e^{\sigma}-1 = f(\sigma) \tag{67}$$

$$\dot{\sigma} = x \qquad \dot{x} = y$$

thus, reduce the original system of eqs. (49-52) to the following

$$\begin{aligned} \dot{\sigma} &= x \\ \dot{x} &= y \\ \dot{y} &= -(g_1+g_2)\, y - F(\sigma)x - \Phi_\infty R_2 f(\sigma) \\ F(\sigma) &= g_1 g_2 + \Phi_\infty R_o e^{\sigma} > 0 \end{aligned} \tag{68}$$

The equilibrium state of this system is $\sigma=x=y=0$ and it is linearly unstable when ρ_o is larger than the upper bound given by (65). The question then is whether there is a bounded region around the equilibrium state beyond which the growth of the reactor parameters is limited. The answer is yes as it can be easily deduced from the following considerations.

First note that for $x < 0$ $(d\Phi/dt < 0)$ and for all σ and y the system is physically bounded. Consequently, it is of interest to examine what happens only for $x > 0$. To this end consider the following functions.

a. $$V_1 = \tfrac{1}{2}\left[y+(g_1+g_2)x+\int_0^{\sigma} F(\sigma)d\sigma\right]^2 + (g_1+g_2)\Phi_\infty R_2 \int_0^{\sigma} f(\sigma)d\sigma \tag{69}$$

This function is positive definite $(V_1(\infty) \rightarrow \infty)$ and its time derivative is:

$$\dot{V}_1 = -\left[y+(g_1+g_2)x + \int_0^{\sigma} F(\sigma)d\sigma\right] + (g_1+g_2)\Phi_\infty R_2 f(\sigma)x =$$

$$= -\Phi_\infty R_2 f(\sigma)\left[y+\int_0^{\sigma} F(\sigma)d\sigma\right] \tag{70}$$

This derivative is negative for all x in the following regions:

$$(\sigma > 0,\ y > 0) \quad ; \quad (\sigma < 0,\ y < 0)$$
$$(\sigma > 0,\ -y < \int_0^\sigma F(\sigma)d\sigma) \quad ; \quad (\sigma < 0,\ y < -\int_0^\sigma F(\sigma)d\sigma) \tag{71}$$

consequently, V_1 is a Liapunov function in these regions and therefore the reactor parameters are bounded for σ and y satisfying conditions (71). This result is schematically shown in fig. 1.

b. $$V_2 = \tfrac{1}{2}\Big[y+\Phi_\infty R_0\int_0^\sigma f(\sigma)d\sigma\Big]^2 + \tfrac{1}{2}(g_1 g_2 + \Phi_\infty R_0)x^2 \tag{72}$$

V_2 is also a positive definite function ($V_2(\infty) \to \infty$) and its derivative is:

$$\dot{V}_2 = -\Big[y+\Phi_\infty R_0 \int_0^\sigma f(\sigma)d\sigma\Big]\Big[(g_1+g_2)y+\Phi_\infty R_2 f(\sigma)\Big] - \\ - \Phi_\infty R_0(g_1 g_2+\Phi_\infty R_0)x\int_0^\sigma f(\sigma)d\sigma \tag{73}$$

Note that for $x > 0$, $\sigma < 0$ and $y > \frac{R_2}{g_1+g_2}$

$\dot{V}_2 < 0$ and consequently the reactor parameters are bounded in the region shown in fig. 2.

c. $$V_3 = \tfrac{1}{2}\Big[y - \frac{R_1}{R_0}x + \int_0^\sigma F(\sigma)d\sigma\Big]^2 \tag{74}$$

$$\dot{V}_3 = -\frac{R_2}{R_0}\Big[y - \frac{R_1}{R_2}x + \int_0^\sigma F(\sigma)d\sigma\Big]\Big[y+\Phi_\infty R_0 f(\sigma)\Big] \tag{75}$$

Note that for $x > 0$, $\sigma > 0$ and $-y > \int_0^\sigma F(\sigma)d\sigma$, $\dot{V}_3 < 0$ because

$$\int_0^\sigma F(\sigma)d\sigma = g_1 g_2\sigma + \Phi_\infty R_0 f(\sigma) > \Phi_\infty R_0 f(\sigma) \tag{76}$$

Therefore, the reactor parameters are bounded in the region shown in fig. 3.

d. $$V_4 = \tfrac{1}{2}\left[y+(g_1+g_2)x\right]^2 + \int_0^\sigma f(\sigma)d\sigma \qquad (77)$$

$$\dot{V}_4 = -\left[y+(g_1+g_2)x\right]\left[F(\sigma)x + \Phi_\infty R_2 f(\sigma)\right]+f(\sigma)x \qquad (78)$$

For $y > 0$, $\sigma < 0$ and $x > \dfrac{\Phi_\infty R_2}{g_1 g_2+\Phi_\infty R_0}$ the derivative $\dot{V}_4$ is negative and therefore the reactor parameters are bounded in the region shown in fig. 4.

Simultaneous consideration of $V_1, \ldots V_4$ reveals that the reactor parameters are bounded everywhere but for a small region around the equilibrium state defined by

$$\sigma < 0; \; x < \frac{\Phi_\infty R_2}{g_1 g_2+\Phi_\infty R_0} \; ; \quad y < \frac{\Phi_\infty R_2}{g_1+g_2} \; ; \; y > -\int_0^\sigma F(\sigma)d\sigma \qquad (79)$$

The sufficient conditions for this to be true are

$$R_2 = -\,(r_1\alpha_1 g_2 + r_2\alpha_2 g_1) > 0; \; R_0 = -\,(r_1\alpha_1 + r_2\alpha_2) > 0 \qquad (80)$$

Consequently under these conditions the system is Lagrange stable. Whether this system is practically acceptable depends on the magnitude of the region defined by inequalities (79).

6. DISCUSSION

The preceeding sections give a brief review of some analytical techniques used in stability investigation of nonlinear systems.

Liapunov's direct method has been emphasized as the most powerful method for this purpose. There are several arguments that support this view.

First, for a large class of systems the method can be readily implemented and sufficient conditions for asymptotic stability can be derived.

Second, for a large number of nonlinear control systems, the requirements for absolute asymptotic stability are the same as the requirements for unconditional linear stability.

Third, in many cases Liapunov's direct method does not require detailed knowledge of the nonlinear effects (see for instance the condition of $(\sigma) > 0$).

Fourth, the method surveys the entire family of solutions of a system of nonlinear ordinary differential equations without resorting to series expansions. This is a great asset since series expansions are always bound to introduce over-restrictive mathematical limitations.

Of course, Liapunov's direct method has also many disadvantages. In many cases the construction of a Liapunov function is extremely difficult if not impossible. In other problems the requirements for asymptotic stability are so severe that they cannot be achieved. A case in point is the boiling water reactor which is linearly unstable when the power is up to a certain level.

In the latter problems the approach to the question of stability from the point of view of Lagrange stability is most appropriate. As it is discussed in section 5, Liapunov's direct method proves again very helpful, if it can be implemented.

The extention of Liapunov's direct method to Lagrange stable nonlinear systems in general and nonlinear reactor dynamics in particular is new. However, it is felt that it will provide a useful tool for further analytical studies.

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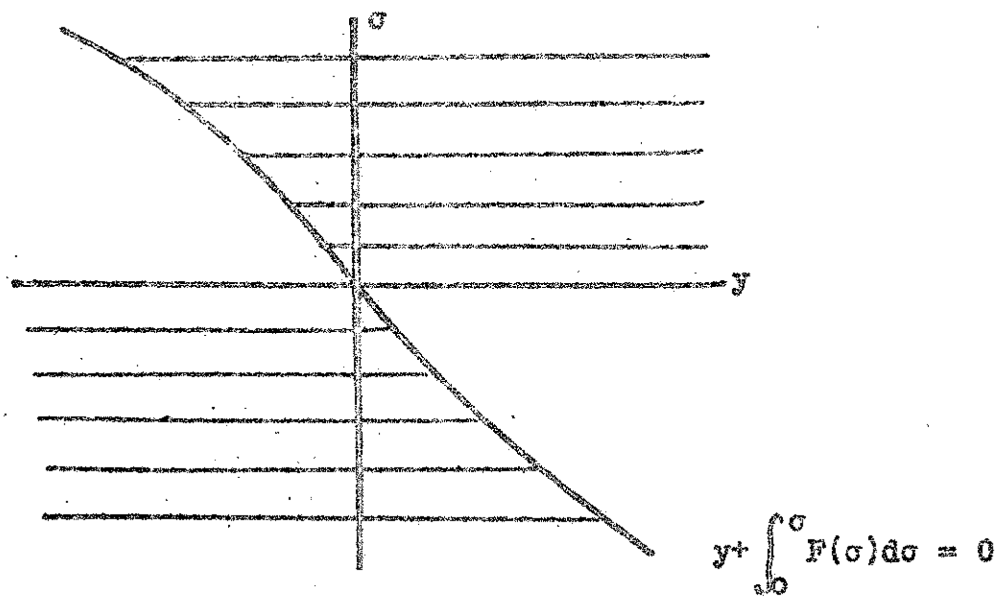

Fig. 1 Stability regions defined by V_1

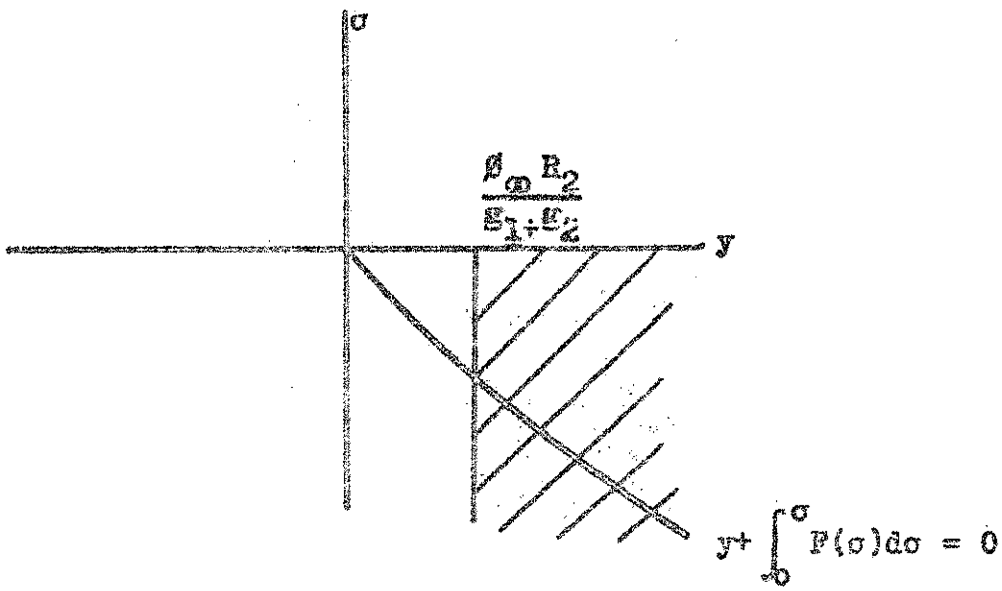

Fig. 2 Stability region defined by V_2

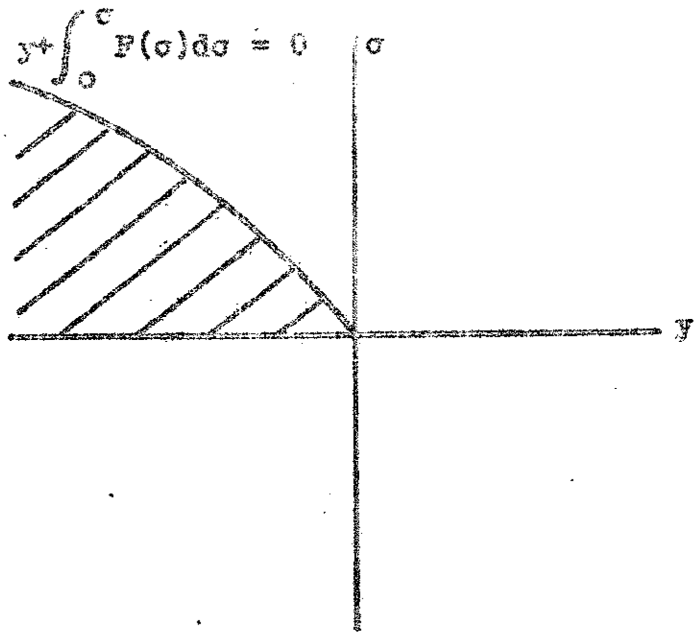

Fig. 3 Stability region defined by V_3

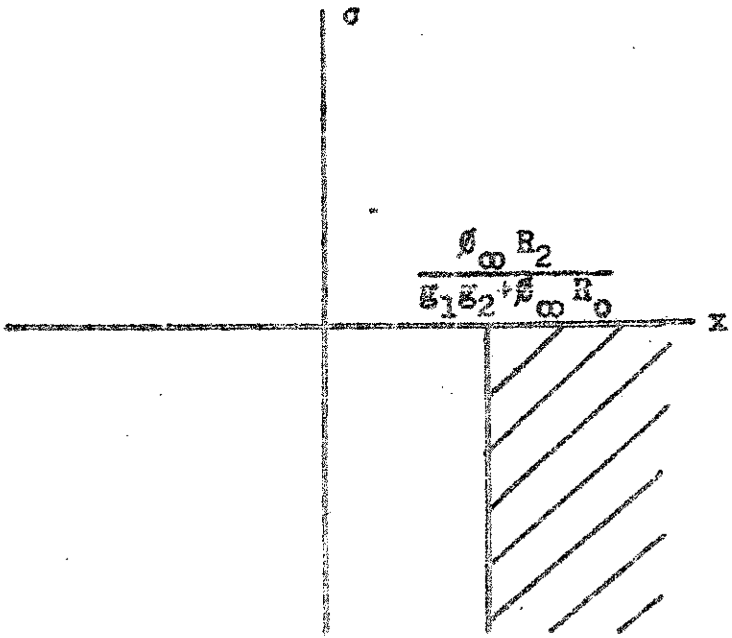

Fig. 4 Stability region defined by V_4