LAGRANGE STABILITY BY LIAPUNOV'S DIRECT METHOD

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INTRODUCTION

The purpose of this paper is to extend Liapunov's direct method[1] to autonomous nonlinear dynamic systems which may be nonstable in the small but which are stable in the large. Such systems are usually referred to as being Lagrange stable.[2] The interest in Lagrange stable systems stems from the following considerations.

Many dynamic systems of electrical, mechanical, or nuclear or of some other nature can be described by a set of n-state variables $x_1,x_2,\ldots,x_n$. These variables can be visualized as the coordinates of a point in an n-dimensional space or equivalently as the components of a vector x in the same space.

Without loss of generality it can be assumed that the point $x = 0$ is an equilibrium, or steady, state. One of the fundamental questions of the theory of system dynamics concerns the type of stability of this equilibrium state. Specifically, if at time $t = 0$ the system is perturbed from its equilibrium $[x(t = 0) \neq 0]$, the question arises as to whether or not as t approaches infinity the variables of the system resume their equilibrium values (the system is asymptotically stable), are bounded (the system is stable), or diverge (the system is unstable).

This question can often be elegantly answered by means of Liapunov's direct method if the dynamics of the system is adequately represented by a set of ordinary differential equations.

In the literature, particularly the Russian literature, there are many problems whose stability has been investigated by means of

Liapunov's direct method.[3–9] In all these problems, however, one finds that the derived sufficient conditions for stability are always conditions for asymptotic stability or stability in the immediate vicinity of the equilibrium state. In addition, it has been shown[10] that for the broad class of nonlinear systems considered by Lur'e[4] and Letov[5] the derived stability criteria require that the linearized version of the system equations belong to a special class and be unconditionally stable. The same requirement of unconditional linear stability is also inherent in Welton's criterion,[11] which is the only criterion available for nuclear reactor systems.

In many applications where certain state variables must be kept within very close tolerances, asymptotic and consequently linear stability is necessary. In other applications, however, designing for asymptotic stability may be an overrestrictive and needless objective. The desired state of a system may be mathematically nonstable; yet the system variables may be so bounded that the system performance is acceptable. For example, many aircraft, nuclear-plants, and other systems exhibit this condition, but their performance is not considered as undesirable. Such systems are classified as Lagrange stable.

In view of these remarks, it is of practical importance to examine whether or not Liapunov's direct method can also be used for the investigation of Lagrange-stable systems. Such an examination is the objective of this paper, which is organized as follows: A brief description of Liapunov's direct method and its extension to Lagrange-stable systems is given first, and then the method is successfully applied to examples taken from the fields of nuclear reactors and nonlinear control systems. For these examples it is shown that, when Lagrange stability is acceptable, the design specifications are more relaxed than they would be if asymptotic stability were required.

LIAPUNOV'S DIRECT METHOD

Consider a physical system whose dynamics is represented by the set of n ordinary nonlinear differential equations

$$\frac{dx_k}{dt} = \dot{x}_k = X_k(x_1, x_2, \ldots, x_n) \qquad k = 1, 2, \ldots, n \tag{1}$$

or the equivalent vector equation

$$\dot{x} = X(x) \tag{2}$$

In addition, suppose that $x = 0$ is an equilibrium state; i.e., $X(0) = 0$. The type of stability of this state can be investigated by means of a positive definite scalar function $V(x)$ with the following properties:

1. $V(x)$ is continuous, together with its first partial derivatives, in a certain open region Ω about the origin.
2. $V(0) = 0$, $V(\infty) = \infty$.
3. Outside the origin and always in Ω, $V(x)$ is positive.
4. If $\dot{V}(x) \leq 0$ (subject to Eq. 2), then $V(x)$ is called a Liapunov function.

With these definitions, Liapunov's main stability theorems are

I. Stability Theorem. If there exists in the neighborhood Ω of the origin a Liapunov function $V(x)$, then the origin is stable for all perturbations lying in Ω.

II. Asymptotic Stability Theorem. If, in addition to the requirements of theorem I, $\dot{V}(x)$ is negative definite, then the stability is asymptotic.

III. Instability Theorem. If $\dot{V}(x) > 0$ and $V(x)$ assumes positive values arbitrarily close to the origin, then the origin is unstable.

There are other variations of Liapunov's main stability theorems. For these, however, as well as the proofs and a general discussion of theorems I to III, the reader is referred to the literature.[1-3] For the purposes of this paper, it is sufficient to note that the application of Liapunov's stability theorems, as they are stated above, requires the consideration of a region Ω that includes the equilibrium state. This requirement implies that the dynamic behavior of the system in the immediate vicinity of the equilibrium state is important.

As already indicated, there are many practical problems in which the exact performance of the system near its equilibrium state is unimportant. It is therefore intriguing to examine whether or not Liapunov's main theorems could be modified in such a way that (1) the vicinity of the equilibrium state is excluded from the definitions of $V(x)$ and $\dot{V}(x)$ and (2) the requirements for Lagrange stability are less restrictive. It can be shown that such a modification is possible if the stability theorem is stated as follows:

IV. Lagrange Stability Theorem. If there exists a positive definite scalar function $V(x)$ that is continuous, together with all its first partial derivatives and with the property that $V(x) \to \infty$ as $\|x\| \to \infty$, and if $\dot{V}(x) \leq -\epsilon < 0$ (subject to Eq. 2) for all x outside some bounded region M surrounding the equilibrium state $x = 0$, then the system variables are ultimately bounded or the system possesses Lagrange stability.

The proof of this theorem is also given in Ref. 2. Some illustrative applications are given in the next section.

APPLICATIONS

Reactor with Two Temperature Coefficients

Consider the step response of a two-region reactor with two temperature coefficients of reactivity.* The normalized dynamic equations[6,12] are

$$\frac{d\sigma}{dt} = \phi_\infty (r_1\alpha_1 T_1 + r_2\alpha_2 T_2) \tag{3}$$

$$\frac{dT_1}{dt} = -g_1 T_1 + f(\sigma) \tag{4}$$

$$\frac{dT_2}{dt} = -g_2 T_2 + f(\sigma) \tag{5}$$

$$f(\sigma) = e^\sigma - 1; \; \sigma f(\sigma) > 0$$

The equilibrium state is $\sigma = T_1 = T_2 = 0$. Welton's criterion or Lur'e's conditions for stability yield that this state is asymptotically stable when

$$r_1\alpha_1 g_1 + r_2\alpha_2 g_2 < 0; \; r_1\alpha_1 g_2 + r_2\alpha_2 g_1 < 0 \tag{6}$$

Elementary considerations reveal that the meaning of the conditions given in inequalities (6) is that the linearized version of Eqs. 3 to 5 be unconditionally stable for all values of ϕ_∞.

It is interesting to examine what happens when conditions (6) are not satisfied and therefore Welton's or Lur'e's sufficient criteria are not constructive. For example, consider the case

$$\begin{aligned} r_1\alpha_1 g_1 + r_2\alpha_2 g_2 &= R_1 > 0 \\ r_1\alpha_1 g_2 + r_2\alpha_2 g_1 &= -R_2 < 0 \\ r_1\alpha_1 + r_2\alpha_2 &= -R_0 < 0 \end{aligned} \tag{7}$$

It can be easily shown that when conditions (7) are satisfied, the linearized version of Eqs. 3 to 5 is conditionally stable. Specifically the reactor is linearly stable when

$$\phi_\infty < \frac{g_1 g_2 (g_1 + g_2)}{R_1} \tag{8}$$

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*This problem has been analyzed by several authors by means of Liapunov's direct method and other techniques. It is being reconsidered here to indicate the difference in specifications for asymptotic and Lagrange stability.

Subsequently it will be shown that the reactor is Lagrange stable for all values of ϕ_∞. To this effect, rearrange the system of Eqs. 3 to 5 into the equivalent system

$$\dot{\sigma} = x \tag{9}$$

$$\dot{x} = y \tag{10}$$

$$\dot{y} = -(g_1 + g_2)y - F(\sigma)\, x - \phi_\infty R_2\, f(\sigma) \tag{11}$$

$$F(\sigma) = g_1 g_2 + \phi_\infty R_0 e^{\sigma} > 0$$

The equilibrium state is also $\sigma = x = y = 0$. The ultimate boundedness of the variables (σ,x,y) and consequently of the original-state variables (σ,T_1,T_2) is deduced from the following considerations.

First note that for $x < 0$ and for all values of σ and y the system is physically bounded; then examine what happens only for $x > 0$. Consider the functions:

(a) $$V_1 = \frac{1}{2}\left[y + (g_1 + g_2)x + \int_0^{\sigma} F(u)\, du\right]^2 + (g_1 + g_2)\phi_\infty R_2 \int_0^{\sigma} f(u)\, du \tag{12}$$

This function is positive definite $[V_1(\infty) \to \infty]$, and its time derivative is

$$\dot{V}_1 = -\phi_\infty R_2\, f(\sigma)\left[y + \int_0^{\sigma} f(u)\, du\right] \tag{13}$$

The derivative is negative for all values of x in the following regions

$$(\sigma > 0,\ y > 0);\ (\sigma < 0,\ y < 0)$$
$$\left[\sigma > 0,\ -y < \int_0^{\sigma} F(u)\, du\right];\ \left[\sigma < 0,\ y < -\int_0^{\sigma} F(u)\, du\right] \tag{14}$$

According to theorem IV the variables σ, x, and y are bounded in the regions determined by the inequalities of (14). This result is schematically shown in Fig. 1.

(b) $$V_2 = \frac{1}{2}\left[y + \phi_\infty R_0 \int_0^{\sigma} f(u)\, du\right]^2 + \frac{1}{2}(g_1 g_2 + \phi_\infty R_0)x^2 \tag{15}$$

$$\dot{V}_2 = -\left[y + \phi_\infty R_0 \int_0^{\sigma} f(u)\, du\right]\left[(g_1 + g_2)y + \phi_\infty R_2\, f(\sigma)\right] - \phi_\infty R_0 (g_1 g_2 + \phi_\infty R_0)x \int_0^{\sigma} f(u)\, du \tag{16}$$

The derivative $\dot{V}_2 < 0$ for $x > 0$, $\sigma < 0$, and $y > \phi_\infty R_2/(g_1 + g_2)$ (see Fig. 2). The reactor variables are bounded in this region.

(c) $$V_3 = \frac{1}{2}\left[y - \frac{R_1}{R_0}x + \int_0^{\sigma} F(u)\, du\right]^2 \tag{17}$$

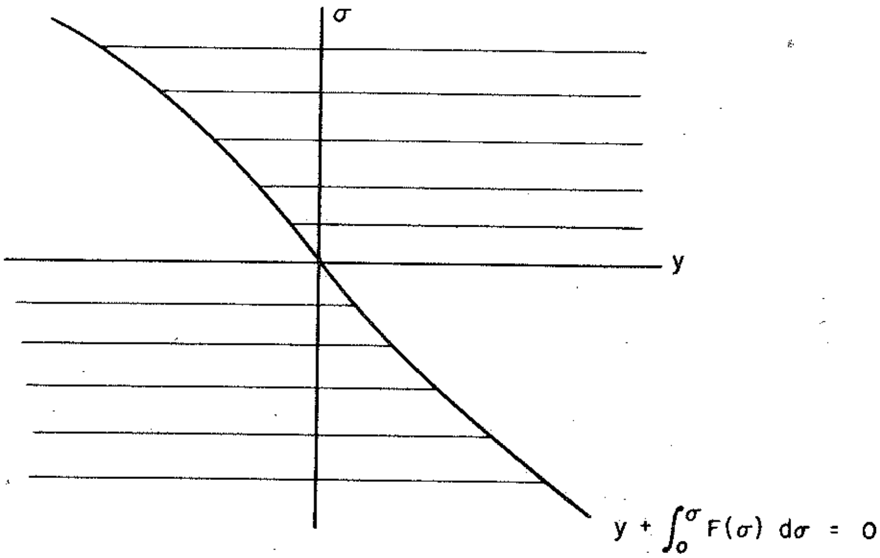

Fig. 1—Stability regions defined by V_1.

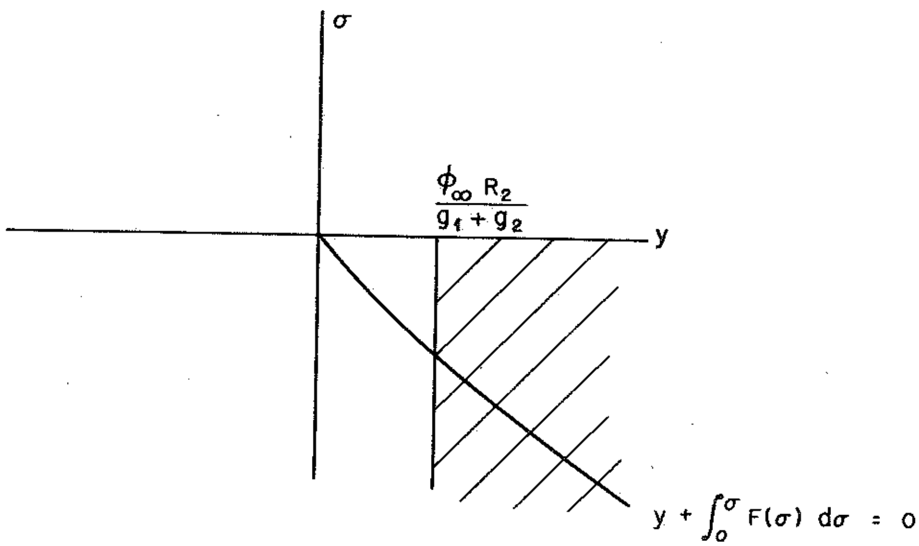

Fig. 2—Stability region defined by V_2.

$$\dot{V}_3 = -\frac{R_2}{R_0}\left[y - \frac{R_1}{R_0}x + \int_0^\sigma F(u)\,du\right][y + \phi_\infty R_0\,f(\sigma)] \tag{18}$$

Note that for $x > 0$, $\sigma > 0$, and $-y > \int_0^\sigma F(u)\,du$ the derivative $\dot{V}_3 < 0$ because

$$\int_0^\sigma F(u)\,du = g_1 g_2 + \phi_\infty R_0\,f(\sigma) > \phi_\infty R_0\,f(\sigma) \tag{19}$$

Therefore the reactor variables are bounded in the region shown in Fig. 3.

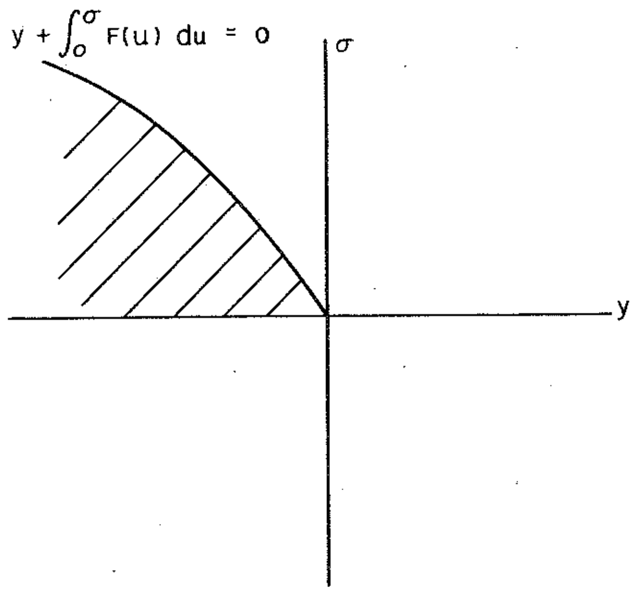

Fig. 3—Stability region defined by V_3.

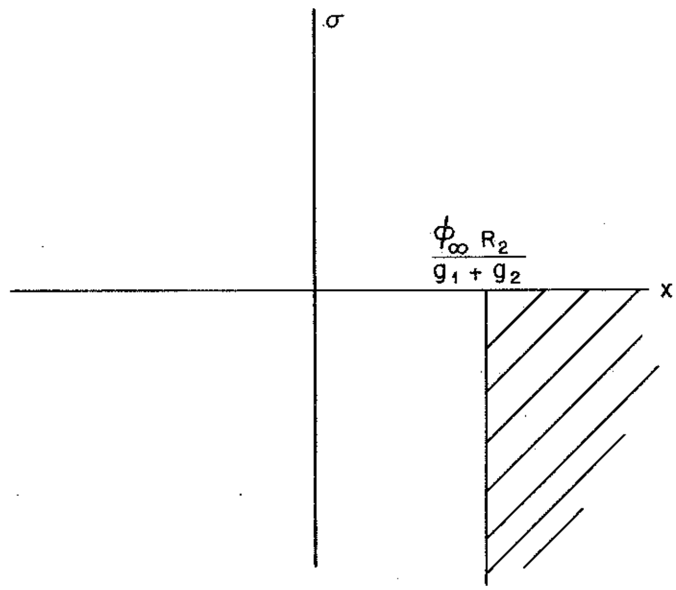

Fig. 4—Stability region defined by V_4.

(d) $$V_4 = \frac{1}{2}[y + (g_1 + g_2)x]^2 + \int_0^\sigma f(u)\,du \tag{20}$$

$$\dot{V}_4 = -[y + (g_1 + g_2)x]\,[F(\sigma)\,x + \phi_\infty R_2\, f(\sigma)] + x\, f(\sigma) \tag{21}$$

For $y > 0$, $\sigma < 0$, and $x > \phi_\infty R_2/g_1 g_2$, the derivative $\dot{V}_4$ is negative and the reactor variables are bounded in the region shown in Fig. 4.

Simultaneous consideration of V_1, V_2, V_3, and V_4 reveals that the reactor variables are ultimately bounded throughout the state space (σ,x,y) except for a region around the equilibrium state defined by

$$\sigma < 0;\ x < \frac{\phi_\infty R_2}{g_1 g_2};\ y < \frac{\phi_\infty R_2}{g_1 + g_2};\ y > -\int_0^\sigma F(u)\,du \tag{22}$$

In other words, the system possesses Lagrange stability when

$$r_1\alpha_1 g_2 + r_2\alpha_2 g_1 < 0;\ r_1\alpha_1 + r_2\alpha_2 < 0$$

Of course, it must be emphasized that whether or not this reactor is of any practical use depends on the size of the region defined by the inequalities of (21).

A General Control System

Consider the general class of systems that have been analyzed by Lur'e and Letov. The dynamics of these systems can be described either by the set of equations

$$\dot{x}_k = \rho_k x_n + f(\sigma) \qquad k = 1,2,\ldots,n \tag{23}$$

$$\dot{\sigma} = \sum_k^n \beta_k x_k - r\, f(\sigma) \tag{24}$$

or by the set of equations

$$\dot{x}_k = r_k x_k + \sigma \qquad k = 1,2,\ldots,n \tag{25}$$

$$\dot{\sigma} = \sum \bar{\beta}_k x_k - \bar{\rho}\sigma - f(\sigma) \tag{26}$$

where x_k and σ are the state variables; $\bar{\rho}$, r, β_k, $\bar{\beta}_k$, ρ_k, and r_k are constant coefficients; $\mathrm{Re}\rho_k$ and $\mathrm{Re} r_k < 0$; and $f(\sigma)$ is a nonlinear function belonging to the class of

(A) $\sigma\, f(\sigma) > 0$ for $|\sigma| > \sigma^*$

$f(\sigma) = 0$ for $|\sigma| \leq \sigma^*$

(A_1) $\sigma\, f(\sigma) > 0$ for all σ

Assume that these systems admit only one equilibrium state, which is $x_1 = x_2 = \ldots = x_n = \sigma = 0$. Lur'e and Letov have proposed a variety of asymptotic stability criteria that they derive by means of Liapunov's direct method. It can be shown that the least restrictive of all these criteria requires that the linearized version of either Eqs. 23 and 24 or Eqs. 25 and 26 be unconditionally stable and of a special class. In particular, if

$$f(\sigma) = h\sigma + \phi(\sigma);\ h \geq 0;\ \sigma\,\phi(\sigma) > 0 \tag{27}$$

the linear approximations of Eqs. 23 and 24 or Eqs. 25 and 26 must have all their eigenvalues in the left half complex plane[10] for all positive values of h. This requirement is very limiting and, in fact, is more restrictive than necessary for the stability of the linear approximation for a given set of values of h. The overrestriction can be overcome if the problem is approached from the standpoint of Lagrangian stability.

For example, consider the system of Eqs. 25 and 26. This system can be transformed into the canonical form

$$\dot{y}_i = \lambda_i y_i - \phi(z) \qquad i = 1,2,\ldots,(n+1) \tag{28}$$

$$z = \sum_{i}^{n+1} k_i y_i \qquad \sum_{i}^{n+1} k_i = 1 \tag{29}$$

where the y_i's are the new canonical variables, the λ_i's are assumed distinct and are the eigenvalues of the linear approximation of Eqs. 25 and 26 for specific values of h, and the k_i's are the elements of the *(n+1)* th column of the inverse of the special transformation matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,n+1} \\ a_{21} & a_{22} & \ldots & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n+1} \\ 1 & 1 & \ldots & 1 \end{bmatrix} \tag{30}$$

Suppose that $\mathrm{Re}\lambda_i < 0$ for all (i). This requirement may be satisfied even when the linear approximation of Eqs. 25 to 26 is not unconditionally stable. In addition, assume that $\phi(z)$ is bounded for all values of $z[|\phi(z)| < C = \text{constant}]$. Then, the positive definite function

$$V = \frac{1}{2}\sum_{i=1}^{n+1} y_i^2 \tag{31}$$

admits a derivative

$$\dot{V} = \sum_{i=1}^{n+1} \lambda_i y_i^2 - \phi(z)\sum_{i=1}^{n+1} y_i < \sum_{i=1}^{n+1} \lambda_i y_i^2 + C\left|\sum_{i=1}^{n+1} y_i\right| \tag{32}$$

This derivative becomes negative when $V > R_0^2$, where R_0^2 is a constant such that the second-order terms in the right-hand side of Eq. 32 predominate over the linear terms. Consequently the system described by Eqs. 28 and 29 or Eqs. 25 and 26 is Lagrange stable even though the linear approximation may not be unconditionally stable, provided that $|\phi(z)| < C$.

Next, suppose that $[z\,\phi(z) \sim z^{2q};\ q > 1]$, that one of the eigenvalues is positive ($\lambda_1 > 0$), and that all the others are in the left-half complex plane $[\mathrm{Re}\lambda_i < 0;\ i=2,3,\ldots,(n+1)]$. This system can never be asymptotically stable. It is, however, Lagrange stable for the following reasons.

First, z is bounded. Indeed,

$$\frac{1}{2}\frac{d}{dt}z^2 = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} k_i k_j \lambda_i y_i y_j - z\,\phi(z) \tag{33}$$

and since $z\,\phi(z)$ varies faster than z^2, there is a value z_0 such that for $|z|>z_0$ the second term in the right-hand side of Eq. 33 predominates over the first and the derivative of z^2 becomes negative.

Second, all the variables $y_2, y_3, \ldots, y_{n+1}$ are bounded because

$$\frac{d}{dt}V_1 = \frac{1}{2}\frac{d}{dt}\sum_{i=2}^{n+1} y_i^2 = \sum_{i=2}^{n+1} \lambda_i y_i^2 - \phi(z)\sum_{i=2}^{n+1} y_i \tag{34}$$

and since z and $\phi(z)$ remain bounded, there is a value R_1^2 such that, when $V_1 > R_1^2$, the first term in the right-hand side of Eq. 34 predominates over the second and the derivative of V_1 becomes negative.

Third, y_1 must be bounded because z and $y_2, y_3, \ldots, y_{n+1}$ are bounded and z is a linear combination of all the y_i's.

This completes the proof that the system described by Eqs. 25 and 26 is Lagrange stable even when the linear approximation is unstable ($\lambda_1 > 0$), provided that the nonlinearity is such that $(z\,\phi(z) \sim z^{2q};\ q > 1)$.

CONCLUSIONS

Liapunov's direct method is used to establish sufficient conditions for Lagrange stability of nonlinear systems around an equilibrium state. These conditions guarantee the ultimate boundedness of the system variables without any concern about the behavior of the system in the immediate vicinity of the equilibrium state and without precluding the possibility of the systems being globally asymptotically stable.

The applications presented indicate that the sufficient requirements for Lagrange stability are less restrictive than the sufficient conditions for global asymptotic stability or local asymptotic stability. This result can be expected to be true in general since away from the equilibrium state the behavior of the system is determined by the nonlinearities.

Whether a Lagrange-stable system is practical or not depends on the size of the region around the equilibrium state outside of which the system variables are bounded. This size indicates the maximum amplitude of possible oscillations that the system may experience. If the amplitude of the oscillations is tolerable compared to the degree of accuracy required of the system variables, then Lagrange stability is just as good as any other type of stability, and it should be accepted since it does not limit the system design so much as the requirements for asymptotic stability.

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