SIGNALS FOR TRANSFER-FUNCTION MEASUREMENTS IN NONLINEAR SYSTEMS

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INTRODUCTION

This paper is concerned with the synthesis of test signals for the measurement of dynamic characteristics of nonlinear systems described by a functional expansion.

The output of a nonlinear system can, in general, be expanded into an infinite sum of functionals of the input.[1–3] Each functional is a multidimensional convolution integral depending upon the input and a kernel that is characteristic of the system. The kernels describe the system in the same sense that the impulse response characterizes a linear system.

The impulse response of a linear system can be measured by a variety of well-known techniques. The measurement of the kernels of a nonlinear system, however, is a more involved problem. Wiener[1] proposed a procedure for measuring the kernels of nonlinear systems by cross correlation between input and output, provided that the input is Gaussian white noise. His experimental procedure, although conceptually simple, is impractical because it necessitates an infinite cross-correlation time.

To avoid the requirement of infinite cross-correlation time and at the same time be able to use the method of Wiener with only minor modifications, it is of interest to devise periodic signals whose correlation functions, over one period and up to some order K, are approximately equal to those of Gaussian white noise. Such signals would require a cross-correlation time of the order of the settling time of the system under investigation, but they would permit the measurement of only a limited number of kernels. It is evident that the finite cross-correlation time is a great advantage, whereas the limitation on the number of kernels that can be measured is of no important concern since in any practical problem one is forced to truncate any infinite series expansion at a finite number of terms.

As a first step in line with the above objective, a family of periodic test signals has been designed which allows the measurement of the kernel $h_1(t)$ in systems whose input $x(t)$ and output $y(t)$ are related by the functional expansion

$$y(t) = \int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau + \int_0^\infty \int_0^\infty h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$
$$+ \text{ higher order functionals containing only an even number of factors } x(t-\tau_i) \qquad (1)$$

Although not all nonlinear systems can be represented exactly by Eq. 1, such a representation is better than the often-assumed linear approximation.

The following sections present the synthesis procedure and the properties of the test signals, describe the experimental setup for the measurement of the kernel $h_1(t)$ in Eq. 1, present some practical applications that require accurate knowledge of $h_1(t)$, and discuss a possible direction in which the work might progress to enable one to measure either $h_1(t)$ in the presence of both even- and odd-order functionals or higher order kernels.

TERNARY PERIODIC TEST SIGNALS

Synthesis Procedure

The family of periodic test signals under consideration includes ternary sequences defined as follows:

1. At any instant of time, the signal may assume only one of three possible normalized values: +1, 0, or −1.
2. The signal is discontinuous and may change value only at event points having uniform spacing Δt.
3. The signal is periodic with period $T = N\,\Delta t$, where $N = 3^n - 1$; n = an integer.
4. The *jth* bit is generated by the linear recursion formula

$$C_j = a_1 C_{j-1} + a_2 C_{j-2} + \ldots + a_n C_{j-n} \qquad (2)$$

where the right-hand side is reduced modulo-3 so that C_j satisfies (1). The coefficients a_i are integers having the value +1, 0, or −1. Given n, signals with period $N\,\Delta t$ exist only for certain sets[4] of coefficients $a_1, a_2, \ldots, a_n$.

Given a proper set of coefficients, $a_1, a_2, \ldots, a_n$, the desired periodic signal is generated by starting with any string of n bits which can be formed from the digits +1, 0, and −1, except the string of n zeroes, and applying the recursion formula until the sequence repeats. A typical signal with $n = 2$ and $a_1 = a_2 = 1$ is shown in Fig. 1.

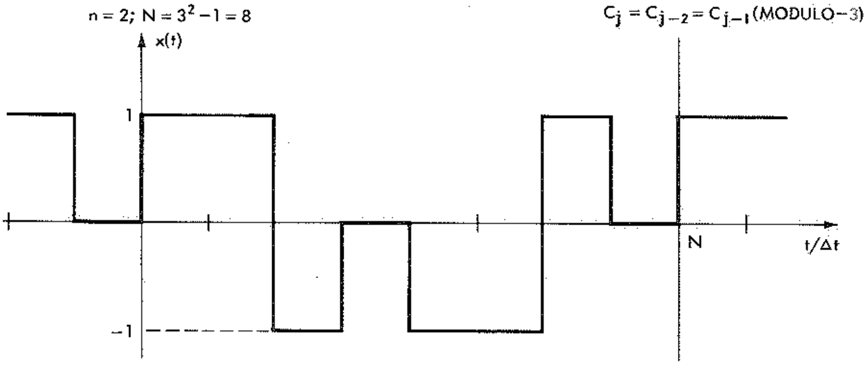

Fig. 1—Typical periodic ternary signal.

Properties of Ternary Periodic Signals

Completeness It is clear from the nature of the recursion formula in Eq. 2 that given n and a suitable set of coefficients, $a_1, \ldots, a_n$, the periodic signal is determined as soon as any n successive bits in the sequence are selected.

The number of strings of n bits that can be formed by allowing each bit in the string to assume independently one of the three possible values, +1, 0, or −1, is 3^n. One of the 3^n possibilities is the string of n zeroes. It is evident, however, that this possibility must be excluded because, if it were considered as an initial string and the recursion formula were applied to it, an endless succession of zeroes would result. Consequently, one period of the repeating sequence cannot contain a number of bits larger than $3^n - 1$. Conversely, those sequences which have the maximum possible number of bits, $3^n - 1$, contain all the n-digit strings that can be formed from the digits +1, 0, and −1, except the string of n zeroes.

Symmetry Given n and a suitable set of coefficients, $a_1, \ldots, a_n$, suppose that one begins to generate a sequence of bits starting with some initial string of digits, $C_1, C_2, \ldots, C_n$, where at least one of the C_i's $\neq 0$. The resulting sequence of period $3^n - 1$ must possess completeness. Next, suppose that one starts with the string $-C_1, -C_2, \ldots, -C_n$. Since the recursion formula is linear, the resulting sequence must be identical with the first except for an inversion of sign; i.e., all +1's and −1's are interchanged with respect to the first sequence. Evidently the second sequence also possesses completeness, and since it is generated by the same recursion formula, it must be a phase-shifted version of the first sequence. In fact, the phase shift is equal to $(3^n - 1)/2$ bits. In other words, in any complete sequence, the second half is derivable from the first by interchanging the +1's and −1's.

Correlation Functions When dealing with periodic functions x(t) of period T, one finds it convenient to define the *kth* order correlation function as

$$\phi_k(\tau_1, \tau_2, \ldots, \tau_k) = \int_{-T/2}^{T/2} x(t)\, x(t+\tau_1) \ldots x(t+\tau_k)\, dt \tag{3}$$

For the present ternary periodic signals, the first-order correlation function, $\phi_1(\tau_1)$, is a periodic function and varies as schematically shown in Fig. 2. It is zero almost everywhere except at alternating

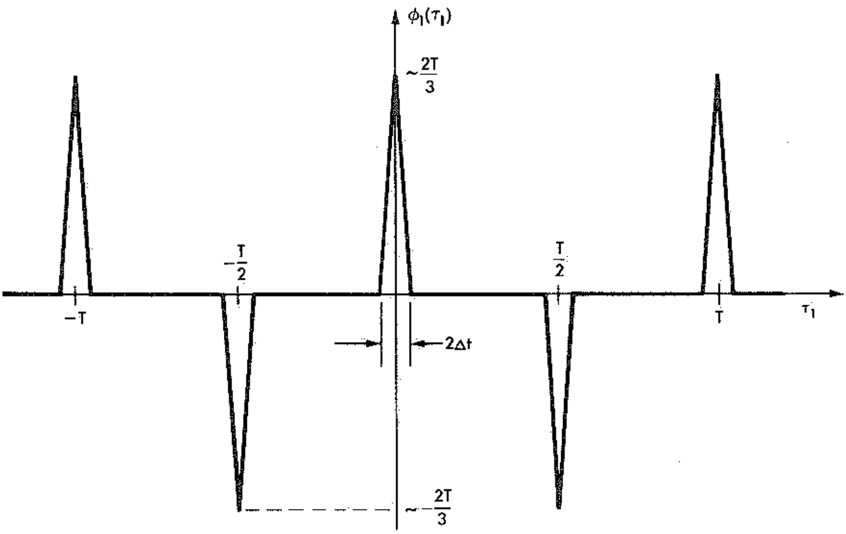

Fig. 2—First-order correlation function of ternary periodic signals.

positive and negative spikes of height $2 \times 3^{n-1}T/(3^n - 1) \cong 2T/3$ for $n \gg 1$ and width $2\Delta t$ spaced $T/2$ apart. It is clear that if n is chosen sufficiently large, the spikes may be made arbitrarily narrow compared to the period T and that the correlation function, $\phi_1(\tau_1)$, may be approximated by the analytical expression

$$\phi_1(\tau_1) = \frac{2T^2}{3^{n+1}} \sum_{-\infty}^{\infty} (-1)^m\, \delta\left(\tau_1 - \frac{mT}{2}\right) \tag{4}$$

The even-order correlation functions are zero everywhere. Indeed, in view of symmetry,

$$\int_{-T/2}^{0} x(t)\, x(t+\tau_1) \ldots x(t+\tau_{2m})\, dt = - \int_{0}^{T/2} x(t)\, x(t+\tau_1) \ldots x(t+\tau_{2m})\, dt \qquad (5)$$

and consequently

$$\phi_{2m}(\tau_1, \tau_2, \ldots, \tau_{2m}) = 0 \qquad (6)$$

EXPERIMENTAL PROCEDURE

Measurement of the Linear Kernel

The measurement of the linear kernel of systems describable by Eq. 1 is achieved by exciting the system with a ternary periodic signal, x(t), and cross correlating the input with the corresponding output. This procedure is similar to the one proposed by Wiener[1] for Gaussian white noise signals and by Balcomb et al.[5] for binary signals. It is schematically shown in Fig. 3.

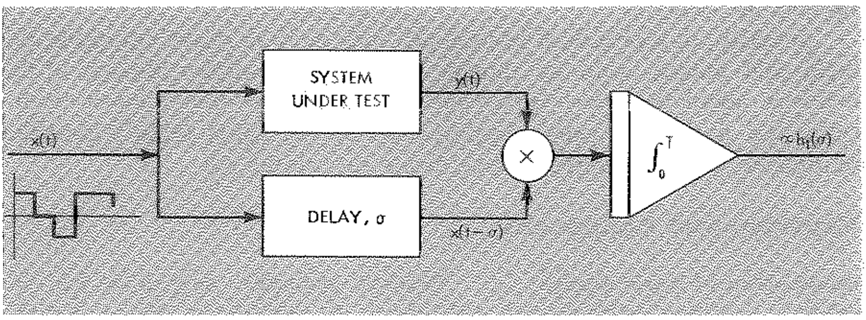

Fig. 3— Experimental arrangement for the measurement of $h_1(\sigma)$.

Indeed, if the cross correlation is performed over one period (Fig. 3), the output of the integrator is

$$\int_0^T y(t)\, x(t-\sigma)\, dt = \int_0^T x(t-\sigma) \Big[\int_0^\infty h_1(\tau)\, x(t-\tau)\, d\tau + \int_0^\infty \int_0^\infty h_2(\tau_1,\tau_2)\, x(t-\tau_1)\, x(t-\tau_2)\, d\tau_1\, d\tau_2$$

$$+ \text{ higher order functionals containing an even number of factors } x(t-\tau_i) \Big]\, dt \qquad (7)$$

If the order of integration is interchanged in the right-hand side of Eq. 7, it is found that

$$\int_0^T y(t)\, x(t-\sigma)\, dt = \int_0^\infty h_1(\tau)\ \phi_1(\tau-\sigma)\, d\tau$$

$$+ \int_0^\infty \int_0^\infty h_2(\tau_1,\tau_2)\ \phi_2(\tau_1-\sigma,\tau_2-\sigma)\, d\tau_1\, d\tau_2$$

$$+ \sum_{m=2} \int_0^\infty \cdots \int_0^\infty h_{2m}(\tau_1, \ldots, \tau_{2m})$$

$$\times\ \phi_{2m}(\tau_1-\sigma, \ldots, \tau_{2m}-\sigma)\, d\tau_1 \ldots d\tau_{2m}$$

$$= \frac{2T^2}{3^{n+1}} \left[h_1(\sigma) + \sum_{m\neq 0} (-1)^m\ h_1\!\left(\frac{\sigma - mT}{2}\right) \right] \tag{8}$$

since $\phi_{2m} = 0$ for all m. Note that if the period, T, of the ternary signal is chosen to be at least twice as long as the settling time* of the linear kernel, $h_1(\tau)$, then the output of the cross correlator is proportional to $h_1(\sigma)$. Consequently, $h_1(\sigma)$ can be measured without appreciable error by varying the delay σ.

Actually, in a practical system one may have to cross correlate the input and output over an integer number of periods because of the inherent noise that may be present in the system. Under these conditions the preceding result remains unaltered, apart from a constant multiplier, and the inherent noise cross correlates out.

Experimental Equipment

Realization of Input Signal Ternary periodic signals can be readily realized with simple equipment. In particular, these signals can be generated with a punched-paper-tape reader with two channels.

One possible method of generating the desired three-level input is shown schematically in Fig. 4. The first channel (A) of the two-channel paper-tape stores the absolute value of the input x(t), which means that the output of this channel is either +1 or 0. This output is applied to a relay-controlled switch. The sign of x(t) is stored in the second channel (B), which means that the output of this channel is positive if x(t) = +1 and negative if x(t) = −1. The output of channel B is applied to the coil of the relay, and the switch is driven either to the +1 or −1 input

*The settling time is defined as the time T_s for which $|h_1(t > T_s)| \leq \epsilon$, where ϵ is an arbitrarily small positive number.

of the summing amplifier. When $x(t) = 0$, the polarity of the coil voltage is immaterial since the input to the amplifier is zero in either case. The desired signal $x(t)$ appears at the output of the amplifier.

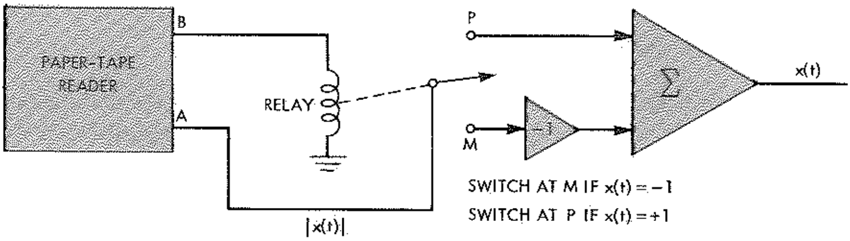

Fig. 4—Realization of ternary test signals.

Cross Correlator The computation of the cross-correlation function between input and output is also readily implemented by a relay-controlled switch. The experimental arrangement is shown in Fig. 5.

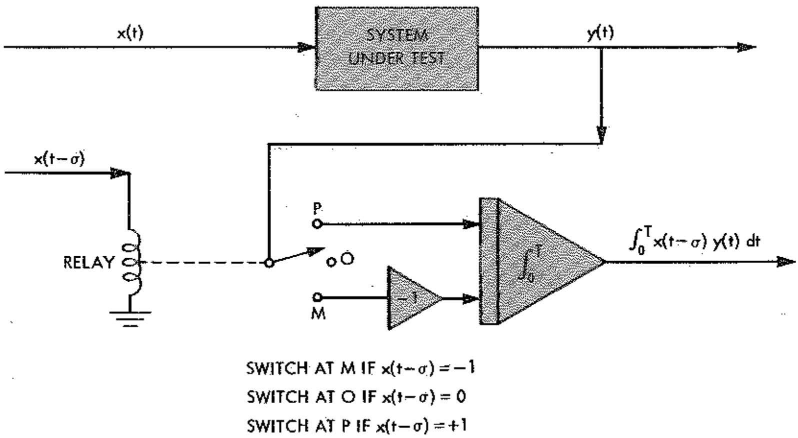

Fig. 5—Analog computation of cross-correlation function.

The output $y(t)$ of the system under test feeds the switch of a three-position relay. The delayed input, $x(t-\sigma)$, which has the value +1, –1, or 0, is used to energize the coil of the relay. The switch is driven to the +1 or –1 input of the integrator or to the open position, depending upon whether the coil voltage $x(t-\sigma)$ is positive, negative, or zero, respectively. Clearly, the input to the integrator is precisely the product $x(t-\sigma)y(t)$; therefore the value of the cross-correlation function for $\tau = \sigma$ appears at the output.

Similar equipment has been used with considerable success by Balcomb et al.,[5] who utilized binary signals to measure the impulse response of reactor systems.

PRACTICAL APPLICATIONS

This paper has been concerned with the problem of determining the linear kernel h_1 in systems whose nonlinearities are characterized by even-order kernels h_2, h_4, etc. Since the nonlinear system is defined by the set of kernels as a whole, and not just by h_1, the question arises as to whether or not there is any practical value in knowing just the linear kernel. Although the answer to this question depends largely upon the nature of the system under investigation, there is a large class of systems in which the linear approximation of the system is of particular interest.

Small Amplitude Response and Stability

The first step in analyzing a nonlinear system is to examine its behavior in the vicinity of its equilibrium state(s). This amounts to truncating the functional expansion of the output after the linear term. The truncation is justified when the input is restricted to appropriately small amplitudes. In addition, the stability properties of the first kernel are indicative of the local and, in some cases, the global stability of the system.[6]

In attempting to measure the linear kernel, one may not be able to treat the system as linear. One reason may be that the presence of inherent noise requires that the amplitude of the input signal be outside the region of validity of the linear approximation. Under these conditions ternary periodic sequences can be used as test input signals without any restrictions on amplitude, provided that the nonlinear system is describable by Eq. 1.

Volterra Integral Equations

There are some nonlinear systems (e.g., the zero-power nuclear reactor) in which the output is given as the solution of a Volterra integral equation of the second kind. Such equations can always be solved by the method of successive approximation, and after some algebra the output y(t) can be written as an infinite sum of functionals of the input x(t). The expansion has the feature that the kernels of the higher order functionals are given explicitly in terms of the linear kernel. It follows that, if the linear kernel can be determined experimentally, all the other kernels will be known as well.

PLANS FOR FUTURE WORK

Work is currently under way on the design of a signal that will allow the measurement of the linear kernel h_1 in systems whose nonlinearities are characterized by a second- and a third-order kernel, h_2 and h_3, respectively. For h_1 to be determined in such systems, a periodic input is desired whose third-order correlation function over one period is a close approximation to that of Gaussian white noise.

The third-order correlation function of Gaussian white noise of spectral density A is given by the expression

$$\phi_3(\tau_1,\tau_2,\tau_3) = A^2 \left[\delta(\tau_1)\,\delta(\tau_2-\tau_3) + \delta(\tau_2)\,\delta(\tau_1-\tau_3) + \delta(\tau_3)\,\delta(\tau_1-\tau_2)\right] \tag{9}$$

This equation says that $\phi_3(\tau_1,\tau_2,\tau_3)$ assumes a peak value when $\tau_1 = \tau_2 = \tau_3 = 0$, that it assumes one-third of that peak value when one of the τ's is equal to zero and the remaining are equal but different from zero, and that it is equal to zero for any other combination of the τ's.

The ternary periodic signals developed previously in this paper have the property that the third-order correlation function $\phi_3(\tau_1,\tau_2,\tau_3)$ has its peak value at (0,0,0) and also the property that

$$\phi_3(0,\tau,\tau) = \frac{2}{3}\,\phi_3(0,0,0) \qquad \tau \neq 0 \tag{10}$$

However, the function ϕ_3 is not zero at all the points at which the corresponding function of Gaussian white noise is zero. An attempt is being made to modify the present ternary signals in order to achieve, if possible, the desired idealized behavior of the third-order correlation function.

It is easily verified that if there is a periodic signal that has a third-order correlation function similar to that of Gaussian white

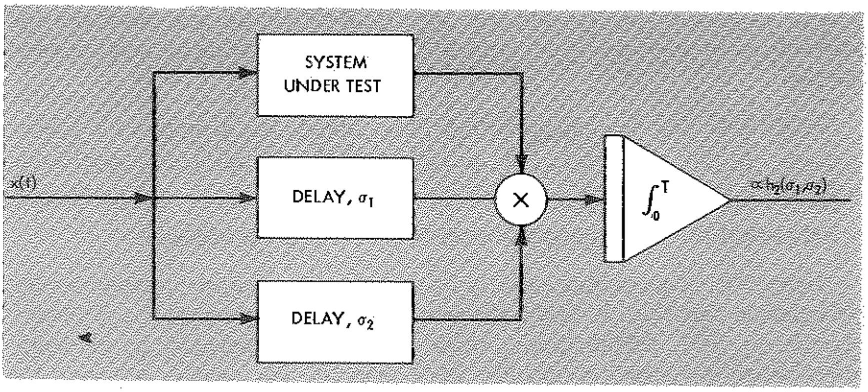

Fig. 6— Experimental arrangement for the measurement of $h_2(\sigma_1,\sigma_2)$.

noise, then it would be possible with a cross correlation to measure also the kernel h_2 in systems characterized by h_1 and h_2. The measurement could be achieved by a cross-correlation method like that shown in Fig. 6.

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DISCUSSION

DOUGLAS BALCOMB (Los Alamos Scientific Laboratory): Have you investigated the properties of the prime chains of the form 4k−1 for their higher order properties?

HOOPER: I haven't looked into the higher order correlation functions of this type of chain.

STEPHEN MARGOLIS (Westinghouse Electric Corporation, Bettis): If one is just trying to measure the first-order kernel, what are the advantages of ternary over binary noise? Just a comparison between the two.

HOOPER: Do you mean simply measuring the kernel of a linear system or do you refer to measuring the kernel h_1 in a nonlinear system?

MARGOLIS: Measuring the kernel in a nonlinear system.

HOOPER: The binary signals do not have the desired second-order correlation function, namely, the one that is zero everywhere. Ternary signals do, and, furthermore, their first-order correlation, or autocorrelation function, is exactly zero between the peaks, whereas for binary signals this function has some small nonzero value.

PETER BENTLEY (United Kingdom Atomic Energy Authority, Risley): Can you tell me please how you decide on the values of a_1, a_2, etc., in forming the chains?

HOOPER: For my own work I have simply generated some chains by trial and error. The article by B. Elspas in the *IRE Transactions on Circuit Theory,* 1959, gives an expression for predicting, given n, the number of chains of length $3^n - 1$ which can be formed. Generally speaking, the longer the chain, the more chains there are of that length. The article by Elspas tabulates, up through $n = 4$, all the sets of coefficients $a_1, a_2, \ldots, a_n$ which generate chains of length $3^n - 1$.