

CHAPTER 3

General Reactor Dynamics

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INTRODUCTION

Designing a safe nuclear plant or maintaining the safety margin of an operating plant which is statically sound requires an effective understanding of the dynamics of the reactor and its associated components and equipment. Such an understanding is necessary in order to ensure that the constraints, imposed either by the materials of the plant components or by the environment in which the plant operates, will not be exceeded at any time.

The constraints that are inherent in various materials in conjunction with the environmental conditions are discussed in other chapters.

*Except for a few changes and additions made in proof, this chapter is based on information in the literature or known to the author prior to November 1963.

The purpose of this chapter is to introduce some general mathematical techniques and experimental procedures that are of import to the problem of understanding the dynamics of nuclear reactors. How these techniques and procedures have been implemented and put to practical use for specific reactors is discussed in the chapters on the kinetics of specific reactor types.

The attempt to understand nuclear reactor dynamics, or for that matter the dynamics of any physical system, involves several closely inter-related steps. The first step is to select a set of variables, the state variables, that are adequate to characterize the physical processes taking place during the operation of the system. Typical nuclear plant state variables are the neutron density, the coolant temperature, the control rod position, etc. The state variables must be directly or indirectly measurable and must be directly or indirectly related to the constraints imposed by the materials and the properties of the environment of the plant. In addition, the state variables must also be chosen so as to preserve a reasonable balance between the desired simplicity of characterization of the system and the required detail of characterization.

It must be emphasized that the problem of selection of state variables is extremely important. From the experimental point of view, the choice of the state variables is influenced by the available instrumentation for the plant since, in principle, all design specifications should be achieved by monitoring and controlling these variables. From the analytical point of view, the selected state variables specify the level of mathematical sophistication necessary for the theoretical visualization of the plant.

The second step in the process of understanding reactor dynamics is to find the time-dependent equations that interrelate the different variables. If all physical processes that partake in the operation of the system are known and if the appropriate state variables have been chosen, this step is relatively easy. The relations between variables are usually based on well-founded conservation laws such as the conservation of energy, mass, elementary particles, momentum, etc. at any instant of time and at every position in the nuclear plant and, therefore, time-dependent balance equations can be written fairly readily.

The third step is to solve the dynamic equations either analytically or with a computer. The objective of this step is either to establish results that can be implemented experimentally, thereby verifying the practicality of the equations, or to predict the range of values of the influential design parameters so that limitations imposed by the constraints can be satisfied. This is a purely mathematical and not a physical problem and it is one of the most difficult. Even for relatively simple systems, the establishment of general solutions is practically impossible. Thus one is forced to consider approximations either in the detail of the characterization of the system or in the range and form of variation of the parameters and variables involved. Approximations lead to practical results provided the results are used in their range of validity. Approximations and the evaluation of their range of validity require a thorough understanding of the

physics of the system.

Finally, the practical value of any and all of the preceding steps must be justified by experiment, as already indicated. Any physical concepts or mathematical models or any analytical or computer results, regardless of their degree of elegance and sophistication, are of no consequence, particularly in questions of nuclear safety, if they have not been or cannot be justified and repeatedly verified by experiment.

The subsequent sections indicate briefly how these four general steps are specifically implemented when nuclear reactor safety is the primary concern.

1 NUCLEAR REACTOR DYNAMICS

1.1 General Remarks

From the standpoint of reactor safety, the most important variable is the energy stored in the reactor and its spatial distribution because it can be directly related to all other variables and constraints. In the final analysis, it is the stored energy, its distribution between fuel, moderator, and coolant and its time rate of change that determine the consequences of a serious accident.

Even though the stored energy is such a key variable in reactor safety, it has not been possible to measure it and control it directly. Instead, the energy behavior is inferred from measurements of the neutron population in the reactor, the temperatures of different components, the pressure, etc. The neutron population is indicative of the time derivative of the energy released in the reactor while temperatures and other thermodynamic variables are measures of the energy stored in the various regions of the reactor.

The analysis of the time behavior of the neutron population in a reactor can be treated by a number of alternative models. The common basis of all these models is that they are analytical statements of the fundamental equality:

$$\left[\begin{array}{c} \text{Rate of change} \\ \text{of neutron} \\ \text{population} \end{array} \right] = \left[\begin{array}{c} \text{Rate} \\ \text{of neutron} \\ \text{production} \end{array} \right] - \left[\begin{array}{c} \text{Rate} \\ \text{of neutron} \\ \text{destruction} \end{array} \right] \quad (1-1)$$

The difference between models lies in the theoretical concepts that are used to express quantitatively the various physical processes that affect the rates of production and destruction. Similar rate equalities can also be used for the other variables which are useful in the study of the behavior of the energy stored in the reactor.

The most general method of analytically implementing Eq. (1-1) is by means of transport theory. This theory allows a precise and detailed representation of all conceivable interactions that neutrons undergo in a reactor with a minimum number of assumptions. The equations that result are very difficult to handle analytically or to implement experimentally because of their generality. Under certain conditions, however, they can be reduced to simpler forms and thus be related in an approxi-

mate way to the energy stored in the reactor. One of these simpler forms is the space-independent conventional system of equations of reactor kinetics, which proved a productive tool in studies of reactor dynamics.

The purpose of this section is to derive the conventional form of reactor kinetics equations starting from the basic notions of transport theory. The value of this approach is that all the factors that influence the dynamics of the reactor are put in evidence, the exact meaning of such convenient parameters as reactivity, neutron lifetime, coefficients of reactivity, etc., can be readily indicated and the assumptions and limitations inherent in the conventional form of reactor kinetics are established. The derivation proceeds along the lines indicated by A. F. Henry [1] with some modifications.

1.2 Neutron Kinetics—Transport Theory

Consider a reactor in which the fuel is stationary. For mathematical simplicity and without any loss of generality, assume that there is only one species of fissionable material and that there is a very large number of neutrons in the reactor. The last assumption implies that statistical variations are ignored.

The time-dependent balance equations for the neutron population and the delayed neutron precursors are:

$$\begin{aligned} \frac{\partial}{\partial t} N(\vec{r}, E, \vec{\Omega}, t) &= \int_{\Omega'} d\vec{\Omega}' \\ &\times \left\{ \frac{1}{4\pi} f_0(E) \int_0^\infty dE' \nu(E') (1 - \beta) \nu' \Sigma_f(\vec{r}, E', t) N(\vec{r}, E', \vec{\Omega}', t) \right. \\ &+ \int_0^\infty dE' \nu' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) N(\vec{r}, E', \vec{\Omega}', t) \left. \right\} \\ &+ \frac{1}{4\pi} \sum_{i=1}^m \lambda_i f_i(E) C_i(\vec{r}, t) + S(\vec{r}, E, \vec{\Omega}, t) \\ &- \vec{\Omega} \cdot \text{grad } N(\vec{r}, E, \vec{\Omega}, t) - \nu \Sigma_t(\vec{r}, E, t) N(\vec{r}, E, \vec{\Omega}, t) \end{aligned} \quad (1-2)$$

$$\begin{aligned} \frac{\partial}{\partial t} C_i(\vec{r}, t) &= \int_{\Omega'} d\vec{\Omega}' \\ &\int_0^\infty dE' \nu(E') \beta_i \nu' \Sigma_f(\vec{r}, E', t) N(\vec{r}, E', \vec{\Omega}', t) - \lambda_i C_i(\vec{r}, t), \\ &i = 1, 2, \dots, m, \end{aligned} \quad (1-3)$$

where $C_i(\vec{r}, t)$

E
 $f_i(E), (i=0, 1, \dots, m)$

$N(\vec{r}, E, \vec{\Omega}, t)$

$S(\vec{r}, E, \vec{\Omega}, t)$

ν

$\nu \vec{\Omega}$
 β_i, β

λ_i

$\nu(E)$

$\Sigma_f(\vec{r}, E, t)$

$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t)$

$\Sigma_t(\vec{r}, E, t)$

delayed neutron precursor concentrations
relative neutron energy
prompt and delayed neutron energy spectrum

neutron density in the $(\vec{r}, E, \vec{\Omega}, t)$ -space
source density in the $(\vec{r}, E, \vec{\Omega}, t)$ -space

neutron speed
neutron velocity
delayed neutron fractions

delayed neutron decay constants

number of neutrons per fission, of energy E
fission cross section
scattering cross section

total cross section

The meaning of Eqs. (1-2) and (1-3) may be understood as follows. Neutron interactions depend on the relative energy and direction of motion of neutrons with respect to the target nuclei. The neutron density, $N(\vec{r}, E, \vec{\Omega}, t)$, is the number of neutrons per unit volume, per unit energy and per unit solid angle that are at the position \vec{r} of the reactor, have a relative energy E and a relative direction of motion $\vec{\Omega}$ at time t . In a small volume ΔV , centered around the point \vec{r} , the time rate of change of neutrons that have energies between E and $E + \Delta E$ and directions of motion between $\vec{\Omega}$ and $\vec{\Omega} + d\vec{\Omega}$ is

$$\Delta V \Delta E \Delta \vec{\Omega} \frac{\partial}{\partial t} N(\vec{r}, E, \vec{\Omega}, t). \quad (1-4)$$

Neutrons are born in the elementary volume $\Delta V \Delta E \Delta \vec{\Omega}$ of the phase space $(\vec{r}, E, \vec{\Omega})$ through the production mechanisms:

prompt fissions,
elastic and inelastic scattering collisions,
decay of delayed neutron precursors, and
external neutron sources.

Thus, the rate of production in $\Delta V \Delta E \Delta \vec{\Omega}$ can be written as:

$$\begin{aligned} &\left[\int_{\Omega'} d\vec{\Omega}' \left\{ \frac{f_0(E)}{4\pi} \int_0^\infty dE' \nu(E') (1 - \beta) \nu' \Sigma_f(\vec{r}, E', t) N(\vec{r}, E', \vec{\Omega}', t) \right. \right. \\ &\quad \left. \left. + \int_0^\infty dE' \nu' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}, t) N(\vec{r}, E', \vec{\Omega}', t) \right\} \right. \\ &\quad \left. + \frac{1}{4\pi} \sum_{i=1}^m \lambda_i f_i(E) C_i(\vec{r}, t) + S(\vec{r}, E, \vec{\Omega}, t) \right] \Delta V \Delta E \Delta \vec{\Omega}, \end{aligned} \quad (1-5)$$

provided that both fission neutrons and delayed neutrons are emitted isotropically and all fission frag-

ments remain in the immediate vicinity of the fission event that produces them. The cross sections are taken explicitly dependent on time in order to account for externally introduced changes in the reactor materials.

On the other hand, neutrons are lost from the elementary volume $\Delta V \Delta E \Delta \bar{\Omega}$ through the destruction mechanisms:

leakage, and

absorption, including scattering-out collisions.

Thus the rate of destruction can be written as

$$\{\bar{\Omega} \nabla \cdot \text{grad } N(\vec{r}, E, \bar{\Omega}, t) +$$

(leakage)

$$+ v \Sigma_t(\vec{r}, E, t) N(\vec{r}, E, \bar{\Omega}, t) \Delta V \Delta E \Delta \bar{\Omega}.$$

(total absorption)

(1-6)

It is evident that the combination of expressions (1-4) to (1-6), as indicated by the principle of conservation of particles, Eq. (1-1), yields the balance Eq. (1-2).

A similar procedure can be used to verify the delayed neutron precursor Eqs. (1-3).

Equations (1-2) and (1-3) are basic and quite general. Their generality, however, renders them extremely difficult to use in this form for practical studies. The difficulties stem both from the dependence of the equations on seven independent variables, $(\vec{r}, E, \bar{\Omega}, t)$, and from the fact that the macroscopic cross sections are complicated implicit functions of the behavior of the neutron population throughout the entire reactor. Indeed, the macroscopic cross sections depend on the densities of the reactor materials. These densities are determined by the energy stored in the reactor which in turn is related to the integral of the neutron density, $N(\vec{r}, E, \bar{\Omega}, t)$, and to the fission cross section, $\Sigma_f(\vec{r}, E, t)$, as well as to the mechanisms that are used to extract energy from the reactor. The exact functional relationship between $N(\vec{r}, E, \bar{\Omega}, t)$ and the cross sections is therefore very involved and it is further complicated by the fact that approximately 10% of the fission energy is associated with long range radiations which may deposit their energy at positions far away from where the fissions occur.

In addition to all these difficulties, it must be recognized that the density $N(\vec{r}, E, \bar{\Omega}, t)$ can not be measured readily by experiment and consequently, even though it is the most general and appropriate attribute of the neutron population in the reactor, its practicality is questionable.

In view of these remarks, it is clear that additional simplifying assumptions and approximations are necessary to reduce Eqs. (1-2) and (1-3) to a form that is more amenable to analysis and experimental interpretation. The reduction can be achieved by a variety of mathematical techniques. Which techniques are appropriate is a question of expediency and the answer depends on the reactor type and the particular aspect of reactor dynamics that is under investigation. For example, series expansions in terms of a complete set of eigenfunctions, perturbation or variational methods,

multi-group calculations, etc., are different techniques that have been used and are more or less suitable for extracting specific information from the transport theory equations [1-5].

Another approach to the problem of simplification of Eqs. (1-2) and (1-3) is to proceed as in the following section.

1.3 Reactor Kinetics—Conventional Form

As already emphasized, the mathematical complexity of the kinetics equations, derived from transport theory, is due to their explicit or implicit dependence on the neutron energy, direction of motion and position. The kinetics equations would be, conceptually at least, easier to handle if they were dependent only on time. Consequently, it is practical to integrate (average) the energy, direction and position variables out of the transport theory equations. The result of the integration is a set of ordinary differential equations with respect to time.

The question is how to perform the averaging. If the dependence of the functions in Eqs. (1-2) and (1-3) on energy, direction and position were completely known (viz. the neutron and delayed neutron precursor densities were known!), then these dependences could be replaced in Eqs. (1-2) and (1-3) and the equations reduced to a set of ordinary integrodifferential equations with respect to time. Of course, this is not the case and one has to approximate the dependences and then substitute them into the equations. The result of this replacement is that the ordinary integrodifferential equations that ensue are also approximate and the degree of approximation is of the same order of magnitude as the order of approximation of the energy, direction and position dependences.

The error in the ordinary equations that are sought, with respect to the error in the approximation of the various undesirable dependences, can be reduced if the unknown functions in Eqs. (1-2) and (1-3) are first multiplied by some weighting factors and then the equations are integrated with respect to energy, direction and position over the entire reactor. The weighting factors are usually taken from the exact solutions of equations that are adjoint to Eqs. (1-2) and (1-3). In many cases the adjoint solutions are not known either and it is necessary to use approximate weighting factors as well as energy, direction and position dependences and then integrate. This integration procedure is called bilinear averaging and it implies that the error in the resulting time-dependent equations is of second order with respect to the errors made in the approximations of the various dependences and weighting factors.

Consider next how the preceding concepts are purposefully used to derive the conventional form of reactor kinetics from the basic equations of transport theory.

To this end, consider the source-free adjoint equation, corresponding to some arbitrary critical state of the reactor:

$$\int_{\bar{\Omega}'} d\bar{\Omega}' \int_0^\infty dE' \left[v' \Sigma_{s0}(\vec{r}, E' \rightarrow E, \bar{\Omega}' \rightarrow \bar{\Omega}) \right]$$

$$\begin{aligned}
& + \frac{1}{4\pi} f_i(E) \nu(E) v' \Sigma_{t0}(\vec{r}, E) \Big] N_0^*(\vec{r}, E, \vec{\Omega}) \\
& + \vec{\Omega} v \cdot \text{grad } N_0^*(\vec{r}, E, \vec{\Omega}) - v \Sigma_t(\vec{r}, E) N_0^*(\vec{r}, E, \vec{\Omega}) = 0,
\end{aligned}
\quad (1-7)$$

where

$$f_i(E) = f_0(E)(1 - \beta) + \sum_{i=1}^m f_i(E) \beta_i.$$

Suppose that Eq. (1-7) can be solved and that the adjoint, steady-state density $N_0^*(\vec{r}, E, \vec{\Omega})$ is a known function of $(\vec{r}, E, \vec{\Omega})$. Express the time-dependent neutron density $N(\vec{r}, E, \vec{\Omega}, t)$ as the product of a time-dependent function and a function whose integral over all $(\vec{r}, E, \vec{\Omega})$ is bounded for all values of time t . Specifically:

$$N(\vec{r}, E, \vec{\Omega}, t) = P(t) N_0(\vec{r}, E, \vec{\Omega}, t), \quad (1-8)$$

$$I = \int_u d\vec{r} \int_0^\infty dE \int_\Omega d\vec{\Omega} N_0(\vec{r}, E, \vec{\Omega}, t) < M = \text{constant}. \quad (1-9)$$

Note that Eqs. (1-8) and (1-9) do not involve any assumptions. They are merely definitions of the functions $P(t)$ and $N_0(\vec{r}, E, \vec{\Omega}, t)$ that are, conceptually at least, easy to implement. The essence of these definitions is that any growth tendencies of $N(\vec{r}, E, \vec{\Omega}, t)$ can be expressed as a function of time, $P(t)$, only and that the shape of $N(\vec{r}, E, \vec{\Omega}, t)$ within the reactor and in the energy and solid-angle spaces can be described by the shape function $N_0(\vec{r}, E, \vec{\Omega}, t)$ which may vary with time but in such a manner that both the function and its integral over $(\vec{r}, E, \vec{\Omega})$ remain finite at all times.

Next substitute Eq. (1-8) into Eqs. (1-2) and (1-3) and multiply both sides of the equations by $N_0^*(\vec{r}, E, \vec{\Omega})$ and both sides of Eq. (1-7) by $P(t) N_0^*(\vec{r}, E, \vec{\Omega}, t)$. Then integrate the resulting equations over the entire volume of the reactor, over all energies and over all solid angles to find:

$$\frac{d}{dt} P(t) = \frac{\rho - \beta}{\Lambda} P(t) + \sum_i \lambda_i C_i(t) + Q(t) - \frac{\partial \ln(\Lambda F)}{\partial t} P(t) \quad (1-10)$$

$$\frac{d}{dt} C_i(t) = (\bar{\beta}_i / \Lambda) P(t) - \lambda_i C_i(t) + [\partial \ln(\Lambda F) / \partial t] C_i(t),$$

$$i = 1, 2, \dots, m, \quad (1-11)$$

where if

$$\int_u du \dots = \int_v d\vec{r} \int_0^\infty dE \int_\Omega d\vec{\Omega} \dots,$$

$$\rho = \rho(t)$$

$$\begin{aligned}
& = \frac{1}{F} \int_u du \left\{ N_0^*(\vec{r}, E, \vec{\Omega}) \right. \\
& \times \left[\frac{f_i(E)}{4\pi} \int_{\Omega'} d\vec{\Omega}' \int_0^\infty dE' \nu(E') v' \Sigma_{t0}(\vec{r}, E', \vec{\Omega}', t) \right. \\
& + \int_{\Omega'} d\vec{\Omega}' \int_0^\infty dE' v' \Sigma_s N_0(\vec{r}, E', \vec{\Omega}', t) \\
& \left. \left. - v \Sigma_t N_0(\vec{r}, E', \vec{\Omega}', t) \right] \right\}
\end{aligned}$$

$$- N_0^*(\vec{r}, E, \vec{\Omega}) \vec{\Omega} v \cdot \text{grad } N_0(\vec{r}, E, \vec{\Omega}, t)$$

$$- N_0(\vec{r}, E, \vec{\Omega}, t) \vec{\Omega} v \cdot \text{grad } N_0^*(\vec{r}, E, \vec{\Omega}) \Big\}. \quad (1-12)$$

$\delta \Sigma_i$ represents the difference between the time-dependent and time-independent cross sections.

$$F = F(t) =$$

$$\begin{aligned}
& = \int_u du N_0^*(\vec{r}, E, \vec{\Omega}) \sum_{i=1}^m \frac{f_i(E)}{4\pi} \times \\
& \int_{\Omega'} d\vec{\Omega}' \int_0^\infty dE' \nu(E') v' \Sigma_{t0}(\vec{r}, E', \vec{\Omega}', t) N_0(\vec{r}, E', \vec{\Omega}', t)
\end{aligned}
\quad (1-13)$$

$$\Lambda = \Lambda(t) = \frac{1}{F} \int_u du N_0^*(\vec{r}, E, \vec{\Omega}) N_0(\vec{r}, E, \vec{\Omega}, t) \quad (1-14)$$

$$\bar{\beta} = \sum_i \bar{\beta}_i. \quad (1-15)$$

$$\bar{\beta}_i = \bar{\beta}_i(t)$$

$$= \frac{\beta_i}{F} \int_u du N_0^*(\vec{r}, E, \vec{\Omega}) \frac{f_i(E)}{4\pi} \int_{\Omega'} d\vec{\Omega}'$$

$$\begin{aligned}
& \times \int_0^\infty dE' \nu(E') v' \Sigma_{t0}(\vec{r}, E', \vec{\Omega}', t) N_0(\vec{r}, E', \vec{\Omega}', t)
\end{aligned}
\quad (1-16)$$

$$\text{FAC}_i(t) = \frac{1}{4\pi} \int_{\Omega} d\Omega N_0^*(\vec{r}, E, \vec{\Omega}) f_i(E) C_i(\vec{r}, t) \quad (1-17)$$

$$\text{FAQ}(t) = \int_{\Omega} d\Omega N_0^*(\vec{r}, E, \vec{\Omega}) S(\vec{r}, E, \vec{\Omega}, t). \quad (1-18)$$

Equations (1-10) and (1-11) can be reduced to a set of ordinary differential equations with respect to time by neglecting the term $[\partial \ln(AF)/\partial t]$. Thus*

$$\frac{dP(t)}{dt} = \frac{\rho - \beta}{\Lambda} P(t) + \sum_i \lambda_i C_i(t) + Q(t) \quad (1-19)$$

$$\frac{dC_i(t)}{dt} = \frac{\beta_i}{\Lambda} P(t) - \lambda_i C_i(t), \quad i = 1, 2, \dots, m. \quad (1-20)$$

These equations are the conventional reactor kinetics equations that were sought. The next question is what physical meaning, if any, can be attributed to the various quantities appearing in these equations?

To answer the question note first that, apart from the omission of the logarithmic term, these equations would be exact if the calculations implied by the formal definitions of ρ , β_i and Λ could be carried out. As already emphasized, in general this is not possible and $N_0^*(\vec{r}, E, \vec{\Omega})$ and $N_0(\vec{r}, E, \vec{\Omega}, t)$ must be approximated. Thus, ρ , β_i and Λ are known only approximately and therefore Eqs. (1-19) and (1-20) are approximate.

One approximation that is often used is to assume that the neutrons in the reactor are describable by a one-group, age-diffusion theory model. Thus, the neutron densities $N_0^*(\vec{r}, E, \vec{\Omega})$ and $N_0(\vec{r}, E, \vec{\Omega}, t)$ become independent of energy and direction. In the calculations of ρ , Λ and β_i the density $N_0(\vec{r}, t)$ is replaced by the steady-state shape function $N_0(\vec{r})$. In the context of these assumptions, the growth function $P(t)$ is interpreted as proportional to the average reactor power and is experimentally associated with the output of a neutron counter in any position around or inside the reactor. The quantity ρ is interpreted as the reactivity and related to the multiplication factor; Λ is taken as the mean prompt or asymptotic neutron generation lifetime and β_i as the effective delayed neutron fractions. The prompt neutron lifetime Λ and the effective fractions β_i become time invariants of the particular reactor under consideration, when the fission cross section is independent of time.

The physical interpretations, however, of P , ρ , Λ and β_i and the invariance of Λ and β_i cannot be carried outside the context of the one-group age-diffusion theory model which is representative of only a few practical situations.

To see this clearly note that the normalization

*Note that the omission of the term $\partial \ln(AF)/\partial t$ can be avoided by suitable redefinition of ρ and λ_i . The redefinitions are $\rho_1 = \rho - \Lambda \partial \ln(AF)/\partial t$ and $\lambda_{i1} = \lambda_i - \partial \ln(AF)/\partial t$.

factor F is arbitrary. Consequently, even though the kinetics equations are approximately independent of F , the quantities ρ , Λ , and β_i which are dependent on F cannot be measured independently of each other and therefore they cannot be considered, in general, as physically meaningful or invariant parameters. The arbitrary character of F and its consequences basically stem from the fact that what constitutes a neutron production or a neutron destruction process is entirely a matter of personal choice. Specifically, the quantity that determines the behavior of the neutron population in the reactor is the difference between the production and destruction processes. This difference can be separated into two parts in an infinite number of ways. To each way of separation corresponds a normalization factor F and different definitions of ρ , Λ and β_i . Therefore, ρ , Λ and β_i cannot be physically meaningful by themselves.

Even if a choice of F is made once and for all, the coefficients β_i are not identical with the corresponding physical quantities that are established from experimental observations of the neutron decay of fission products. Their values depend not only on the particular reactor under consideration but also on the prevailing operating conditions. They may differ from the physical quantities by 20 to 30%.

Given a choice of F , the conventional equations could be exact if

$$\frac{\partial}{\partial t} (AF) = 0$$

or

$$\frac{\partial}{\partial t} \int_{\Omega} d\Omega N_0^*(\vec{r}, E, \vec{\Omega}) N_0(\vec{r}, E, \vec{\Omega}, t) = 0 \quad (1-21)$$

and ρ , Λ , and β_i can be calculated exactly. Equation (1-21) is true only when the time dependence of the neutron density $N(\vec{r}, E, \vec{\Omega}, t)$ is truly separable from the other variables. Separability of variables is possible only when the reactivity is constant, the source term is negligible and the reactor is on an asymptotic period. Under these conditions β_i and Λ are also invariant and exactly calculable if the neutron spectrum with respect to the reactor materials and the neutron dependence on direction are known. Under these conditions the growth function $P(t)$ is representative of the average reactor power in the sense that the output of any counter at any position of the reactor is proportional to the average reactor power.

Under all other operating conditions, the conventional equations are approximate because the shape function $N_0(\vec{r}, E, \vec{\Omega}, t)$ and the exact variation of the cross sections are not known and they must be approximated for the calculation of the bilinear averages ρ , Λ , and β_i , and because the term $\partial \ln(AF)/\partial t$ is omitted.

In view of the preceding remarks, it is important to examine the value of the conventional equations for analytical studies and for comparisons of theory with experiment. By analytical studies it is meant studies of the solutions, or the properties of the solutions, of equations of the form (1-19) and (1-20)

without specific reference to any particular reactor or to any physical requirements and limitations for experimentally achieving these solutions or properties.

For general analytical studies, the conventional equations are extremely useful. To see this clearly, consider the coefficients of $P(t)$ in Eqs. (1-19) and (1-20) as arbitrary functionals of time and the reactor state variables. If the reactor is not increasingly forced externally, these functionals must remain finite or bounded at all times since the shape function is selected so that all growth tendencies of the neutron density are represented by $P(t)$. Physically, this assumption of boundedness is justified because the growth of the neutron density of a chain reacting system is governed by the number of excess neutrons per neutron causing fission. This number can never be, on the average, greater than ~ 1.5 and, therefore, no matter what the functional dependence of the coefficients of $P(t)$ in Eqs. (1-19) and (1-20) or (1-10) and (1-11) are, these coefficients must be ultimately bounded functions of time. Thus, by assigning to the coefficients a broad variety of reasonable bounded functional representations it is possible to explore the general properties of $P(t)$ that result as solutions of the differential Eqs. (1-19) and (1-20) and thus establish many general trends of reactor dynamics, such as boundedness, stability, and overall transient response. These general trends are extremely useful because they provide the reactor designer or operator with a better understanding of the time behavior of a large variety of reactor operations.

The value of the conventional kinetics equations for comparisons between theory and experiment, however, cannot be qualified in general but must be assessed in the context of the particular experimental conditions under consideration.

When the variations of the state variables of the reactor are small, then β_i and Λ can be approximated by constant values derived from asymptotic calculations (for example, two-group theory) and ρ can be approximated by first-order approximations of the four-factor formula. The growth factor is interpreted as average power and associated with the output of a counter around or in the reactor. Under these conditions good agreement between theory and experiment is established. For many reactor types, reactivity changes of a few tenths of β or less fall into this category. The linearized version of the kinetics equations is an adequate representation of the dynamics of the reactor, and the behavior and distribution of the energy stored in the reactor can be predicted to a satisfactory approximation and correlated with experiment. This is the reason why oscillation, autocorrelation or crosscorrelation analysis and tests, which are discussed in Sec. 3, have proved such productive concepts and tools in the study of reactor dynamics.

When the variation of the state variables of the reactor is large, then the physical meaning of the conventional equations is beclouded and comparison of theory and experiment consists more of semi-empirical fitting of experimental results than of good theoretical predictions of experimental data, particularly if the experimental data are in the form of time traces of the outputs of one or more counters around or inside the reactor. This is due to at

least two facts.

First, the error introduced through the omission of the term $\partial \ln(\Lambda F)/\partial t$, if this term is omitted, is in general of the same order of magnitude as the error involved in approximating the nonlinear Eqs. (1-19) and (1-20) by a set of linear equations with constant coefficients [5]. This error becomes particularly important during fast transients such as the self-limiting, short period SPERT excursions.

Second, all the quantities involved in Eqs. (1-19) and (1-20) cannot be assigned exact physical meanings. For example, suppose that $P(t)$ is interpreted as indicative of the average power of the reactor and it is associated with the record of the output of a counter. Any correlation of experimental and theoretical values during severe transients, where large variations of state variables are involved would be at best fortuitous. Experimentally, two identical counter records may arise from entirely different neutron density distributions as schematically shown in Fig. 1-1. Theoretically, different neutron density distributions lead to different values of ρ , Λ and β_i and different solutions $P(t)$. Hence the difficulty of assuming that $P(t)$ is the average reactor power.

For large perturbations of the state variables, it is more appropriate to abandon the concepts of the conventional kinetics equations and work with other types of approximations of Eqs. (1-2) and (1-3) which usually require elaborate computer codes. Not much work has been done in this area. However, the space-time flux synthesis procedures developed by Kaplan [6] and the space-time solution of a two-group, diffusion approximation of Eqs. (1-2) and (1-3) for boiling water reactors discussed by Wolfe and Greebler [7] are essential steps in the right direction. (See also the discussion of the STAB codes in Sec. 5.3 of the chapter on the Kinetics of Solid-Moderator Reactors.)

In spite of the aforementioned difficulties, the conventional kinetics equations are often used to correlate experimental data pertinent to large excursions. In these correlations, Λ and β_i are taken as invariants, ρ is expressed in terms of some adjustable constants and as a function of power or stored energy, $P(t)$ is interpreted as the power and associated with the output of a counter. The correlations are more or less successful because the many uncertainties involved in the calculations permit the necessary adjustment of the constants to make theory fit the experimental data. Of course,

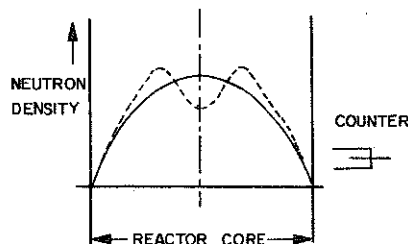


FIG. 1-1 Two different neutron density distributions result in identical counter records. The counter senses only neutron density in peripheral region where distributions coincide.

the subject is by no means well understood yet and more experimentation is needed to permit the establishment of correlations that might be valid for different classes of reactors. There is no doubt that, from a practical standpoint, good semi-empirical correlations are as useful as pure theoretical results.

For a detailed discussion of the use of the kinetics equations in the interpretation of experiments the reader is referred to reference [1].

1.4 Prompt Neutron Lifetime

As already indicated, the prompt neutron lifetime is a concept that is physically meaningful only when the reactor is on an asymptotic period. (For a detailed discussion see also reference [8]). It can then be measured by a variety of small perturbation techniques such as $1/v$ -poisons, oscillation tests, statistical correlations of neutron fluctuations or pulsed neutron experiments. The values of Λ that are thus established range from milliseconds for heavy water reactors to fractions of a microsecond for fast assemblies.

Even though the prompt neutron lifetime is variable during severe transients and it affects the kinetics equations only in terms of the ratios ρ/Λ and β_1/Λ and not by itself, it is interesting to examine what significance may be assigned to the value of the prompt lifetime derived by means of small perturbation techniques and how this value influences the problems of reactor control and safety.

From the standpoint of controllability during normal reactor operations, the magnitude of the prompt neutron lifetime is unimportant. The important reactor time constant, in this regime of normal reactor operation, is determined by the effective delayed neutron precursor constants β_1 and λ_1 and it is several orders of magnitude longer than Λ . Typical values of this time constant are of the order of tenths of a second. Consequently, in this regard, fast, intermediate and thermal reactors present the same operational control problems.

From the standpoint of the power level beyond which stability with respect to small perturbations is lost, the prompt neutron lifetime may be important. For example, for reactors with relatively long heat transfer time constants, such as oxide fuel reactors, the smaller the value of Λ the higher the value of the power level at which the reactor dynamic behavior becomes unstable with respect to small perturbations [9]. Thus, everything else being equal, a reactor with a short prompt neutron lifetime has a higher upper limit of the power level for linear stability than a reactor with a long prompt neutron lifetime. It should be pointed out, however, that instability levels derived from considerations of small perturbations do not necessarily imply that the reactor variables will grow without limit (see also Sec. 2).

From the standpoint of ultimate reactor safety, it is not clear whether the value of the prompt neutron lifetime can be assigned any particular significance or not. Whether a short or long prompt neutron lifetime is the most desirable depends on what is considered to be the most important design criterion. For example, if a reactor is to be controlled externally, say by control rods, then there

is no doubt that, for large changes of ρ compared to β ($\rho/\beta > 1$), a smaller value of the prompt neutron lifetime leads to a more difficult problem of controlling or shutting down the reactor. In this regard, thermal reactors are better than fast reactors. On the other hand, if the reactor has an inherent physical safeguard mechanism, then the value of the prompt neutron lifetime may or may not be relevant to the ultimate safety of the reactor. Specifically, given an inherent shutdown mechanism, the significant safety criterion is the total energy, at the end of a self-limited power burst, stored in the region or regions that introduce the shutdown reactivity effect. If the shutdown reactivity effect is of the form $[-\alpha E^n(t)]$ or $[\alpha E(t) - bE^2(t)]$, where $E(t)$ the stored energy as a function of time and α, b, n constants, then the total energy stored at the end of the burst is independent of the value of the prompt neutron lifetime. In other words, reactors with the same constants (α, n) and the same admissible total energy storage can safely withstand the same amount of reactivity, regardless of whether they are thermal or fast. If, however, the shutdown reactivity effect is of the form $[-\alpha E^n(t-\tau)]$, where τ is a pure time delay between the release of energy in the reactor and the appearance of energy storage in the region that introduces the shutdown reactivity effect, then the total energy stored at the end of a power burst is larger the shorter the prompt neutron lifetime. These general results are discussed in more detail in the chapter on Mathematical Models of Fast Transients.

Finally, from the standpoint of possible assembling rates the prompt neutron lifetime may be used as a measure of the maximum assembling rate that can be tolerated. It turns out that, given an energy-dependent shutdown mechanism, the shorter the asymptotic prompt neutron lifetime the higher the maximum tolerable assembling rate. A qualitative discussion of this result is also given in the chapter on Fast Transients. In addition, a discussion of the role of prompt neutron lifetime in fast reactor accidents is given in the chapter on Fast Reactor Kinetics.

In closing this discussion it must be emphasized that the value of Λ derived by small perturbation techniques is not unique. The lack of uniqueness stems from the fact that each reactor can be critical for a large variety of combinations of its state variables and the reactor constituents. To each critical state corresponds a different prompt neutron lifetime. Which value is more appropriate, or a better approximation for a given problem, depends on the particular type of experiment that one tries to interpret theoretically. This question is discussed in reference [1].

1.5 Reactivity

Regardless of whether β_1 and Λ are approximated by some constant values or are considered as variables, the other quantity that determines the time behavior of $P(t)$, the growth factor of the neutron density, is the reactivity ρ that is defined by Eq. (1-12). When ρ is equal to zero* then there

*Strictly speaking it must be required that $\rho_1 = \rho - \Lambda \partial \ln(\Lambda F) / \partial t = 0$.

are no time-dependent changes in the reactor, or in other words the reactor is critical because

$$\frac{d}{dt} \left[P(t) + \sum_i^m C_i(t) \right] = 0. \quad (1-22)$$

When ρ is different from zero, then $P(t)$ also varies as a function of time.

It is of interest to examine the physical mechanisms that are responsible for the changes of the reactivity ρ and how these changes can be approximated or exactly calculated in terms of measurable reactor state variables.

Inspection of Eq. (1-12) which is the formal definition of reactivity, indicates that the changes in reactivity with respect to an arbitrary critical reference level can be stimulated by the following factors:

- variations in the neutron relative energy spectrum,
- variations of the macroscopic fission, absorption and scattering cross sections, and
- variations of the spatial neutron distribution.

These factors are in turn dependent on variations of the material composition of the reactor, such as variations in fuel, moderator, coolant and structure, and variations of the energy stored in these materials. The dependence of reactivity on the composition of the materials of the reactor stems from the fact that these materials determine the microscopic cross sections that must be used in Eq. (1-12). The dependence of reactivity on the energy stored in these materials is introduced by the fact that this energy determines the thermal agitation and the thermodynamic state of the various target nuclei. Changes in the thermal agitation alter the relative energy spectrum of the reacting neutrons and changes of the thermodynamic state result in variations of the relative target nuclei volume densities which in turn result in variations of the macroscopic cross sections.

Attempts to compute reactivity from Eq. (1-12) lead immediately to the same difficulties that earlier forced the restatement of the transport theory equations in the form of the conventional kinetics equations. In fact, the quantity ρ lumps together in one parameter a major fraction of the computational weaknesses. There are, however, several steps that can be taken which yield very useful practical results without excessive errors.

As a first step, it is found expedient not to compute the reactivity over the entire reactor at once and not to consider simultaneously all the factors that stimulate reactivity changes. Instead, attempts are made to calculate reactivity changes either over distinct reactor regions such as the fuel, the coolant or the moderator or over distinct physical processes such as spectral shifts, void changes, geometric changes, etc., and then to superimpose the different reactivity changes that are so computed. The tacit assumption behind such computational schemes is that the various reactor regions or physical processes that contribute to reactivity are weakly coupled to each other as far as reactivity changes are concerned. This assumption is justified for small changes of reactivity but may lead to intolerable errors for large changes of reactivity.

In the context of this approach to the problem of calculation of reactivity, the various types of reactivity changes are classified in terms of the particular region or physical process under consideration. Thus, terms are used like the fuel, the moderator, or the void reactivity effect, etc.

Even with this expedient isolation of the various contributions to reactivity, however, the computational difficulties are not over. The reason is that the energy stored in the various reactor regions is not readily measurable experimentally and analytically its time-dependent behavior is governed not only by the nuclear interactions but also by the various heat transmission processes that partake in the operation of the reactor. As a second step in the calculation and in order to avoid these difficulties, it is found practical not to relate the previously mentioned reactivity changes or effects directly to the energy stored in the various reactor regions but to other equivalent and measurable thermodynamic variables. Such thermodynamic variables are the fuel or coolant temperature, the void concentration, the coolant pressure or flow, etc.

This approach to the problem of calculation of reactivity introduces another possible classification of reactivity changes in terms of thermodynamic variables. For example, terms are used like fuel temperature reactivity, moderator temperature reactivity, void reactivity, etc.

The relation between measurable thermodynamic variables and corresponding reactivity changes is nonlinear. For small relative changes of the thermodynamic variables, however, the nonlinear relation between a particular variable and reactivity can be approximated by a linear relation such that the reactivity change in question $\rho_i(t)$ takes the form

$$\rho_i(t) = \alpha_i \theta_i(t), \quad (1-23)$$

where $\theta_i(t)$ is a measure of the change of the thermodynamic variable characteristic of the particular region or process with respect to the critical reference reactor and α_i is a constant of proportionality. The constants of proportionality α_i are classified as coefficients of reactivity and thus terms are used like the fuel temperature coefficient of reactivity, the void coefficient of reactivity, the Doppler coefficient of reactivity, etc.

For large changes of the thermodynamic variables, the coefficients of reactivity cannot be taken as constant. They are functions of all the thermodynamic variables. This is another way of saying that the various reactivity changes are coupled to each other. In practice, however, the coefficients of reactivity are often approximated by functions that depend on only one variable. Thus, for large reactivity changes Eq. (1-23) becomes:

$$\rho_i(t) = \alpha_i(\theta_i(t)) \theta_i(t). \quad (1-23a)$$

Finally, the third step in the calculation of reactivity is to establish a relationship between the measures of the change of the thermodynamic variables $\theta_i(t)$ and the energy stored in the reactor or, what is equivalent, the difference between the fission energy and the energy carried away by the coolant. This is also a difficult task involving com-

plicated computations. In practice, the $\theta_i(t)$ are usually expressed in terms of $P(t)$. To this end, the energy balance equations are written for each position of the reactor. These equations are, in general, partial integrodifferential equations similar to Eqs. (1-2) and (1-3) and the thermodynamic variables appear as functions of both space and time. The bilinear averaging procedure, described in Sec. 1.3, can be used to reduce the energy-balance equations into a set of ordinary, time-dependent differential equations of the form:

$$\frac{d\theta_i(t)}{dt} = f_i(P(t), \theta_j(t)), \quad i, j = 1, 2, \dots, n, \quad (1-24)$$

where n is the number of the thermodynamic variables appearing in the energy-balance equations and $\theta_i(t)$ are the averaged measures of the change of these variables. In other words, in principle, by solving the set of Eqs. (1-24), the $\theta_i(t)$ can be expressed as functionals of $P(t)$ which in turn is related to the energy stored in the reactor.

The functions $f_i(P(t), \theta_j(t))$ are, in general, nonlinear. For practical purposes they are often approximated by first order or linear equations and thus:

$$\frac{d\theta_i}{dt} = a_i(P(t) - P_0) + \sum_j b_{ij}\theta_j(t), \quad i = 1, 2, \dots, n, \quad (1-25)$$

where a_i and b_{ij} are constants. Through an appropriate transformation of variables, the set of Eqs. (1-25) can always be transformed into a canonical form such that all the thermodynamic variables are decoupled. Specifically, if the new variables are denoted by $\theta'_i(t)$, then Eqs. (1-25) can always be transformed into the form:*

$$\frac{d\theta'_i(t)}{dt} = a'_i(P(t) - P_0) - g'_i\theta'_i(t), \quad i = 1, 2, \dots, n, \quad (1-26)$$

where a'_i and g'_i are new constants resulting from a_i and b_{ij} through the performed transformation of variables. The reason for casting Eqs. (1-25) into the form of Eqs. (1-26) is that the latter provide another very important classification of reactivity changes in terms of the constants g'_i . If g'_i is large ($1/g'_i$ small), then the reactivity change associated with θ'_i is classified as "prompt." If g'_i is small ($1/g'_i$ is large), then the reactivity change associated with θ'_i is classified as "delayed." As it will become apparent in Sec. 2, the promptness or delayedness of a reactivity change plays a very important role in the dynamic behavior of the reactor.

The justification for the classification of reactivity as prompt or delayed depending on whether g'_i is large or small, respectively, can be understood by considering a step change $P_1 - P_0$, of $P(t) - P_0$. Indeed then, the solution of Eqs. (1-26) is

$$\theta'_i(t) = a'_i(P_1 - P_0)t \cdot \frac{1 - e^{-g'_i t}}{g'_i t}; \quad i = 1, 2, \dots, n. \quad (1-27)$$

If g'_i is large, then the step change contributes very rapidly to the change of $\theta'_i(t)$ and therefore to the reactivity change $\alpha'_i\theta'_i(t)$ and hence the classification of this reactivity as prompt is appropriate. On the other hand, if g'_i is small, the step change requires a relatively long time to be felt by $\theta'_i(t)$ and the corresponding reactivity contribution and hence the classification of this reactivity as delayed. In an asymptotic experiment where $P(t)$ represents the true average power, the term $a'_i(P_1 - P_0)t$ in Eq. (1-27) is the average energy stored in the i th region and the promptness or delayedness of a reactivity change can be thought of as a measure of the effectiveness of the stored energy to produce a sizable reactivity change in a relatively short or long time.

As a reference measure of the magnitude of the inverse time constants g'_i , it is customary to use some value of the prompt neutron lifetime. When $(1/g'_i)$ is smaller than or comparable to the prompt neutron lifetime, the corresponding reactivity change is classified as prompt. When $(1/g'_i)$ is much larger than the prompt neutron lifetime then the corresponding reactivity change is classified as delayed.

It is clear that the values of the inverse time constants g'_i are not unique but depend on the operating conditions prevailing in the reactor. Of particular importance, in this respect, are the heat transfer processes that carry the heat produced by fission away from the reactor.

For example, if there is good heat transfer between the fuel and the coolant, then reactivity changes due to the coolant are relatively prompt. On the other hand, if the heat transfer between the fuel and the coolant deteriorates as the power increases, then the coolant reactivity changes become delayed. Similar remarks can be made about the fuel. For example, if there is heat transfer from the fuel to the coolant the fuel has a certain time constant. If, however, the heat transfer is interrupted either by failures in the fuel-cladding interface or by the blanketing of the fuel-coolant interface with a poor heat conduction layer, then all the fission energy remains in the fuel and the fuel reactivity changes are felt immediately; the fuel time constant becomes much smaller than the prompt neutron lifetime.

In summary, the difficulties involved in the calculation of the reactivity given by Eq. (1-12) make it necessary to impose various simplifications and approximations. These simplifications lead to the concepts of various types of reactivity changes or reactivity effects and coefficients of reactivity. It is clear that the separation of reactivity into the different components is completely arbitrary. This is the reason why different reactor designers characterize their reactors by a variety of reactivity coefficients and effects.

A detailed description of the various reactivity effects and coefficients and specific procedures for their computation in terms of the thermodynamic variables is given in the chapters on the Reactor

*Note that the same transformation of variables must also be used in the expressions for reactivity changes. Thus, in the linear approximation, new coefficients of reactivity α'_i , with respect to the new variables $\theta'_i(t)$, will result.

Core and the Doppler Coefficient and the chapters on the kinetics of different reactor types.

Even though the computation of reactivity is facilitated by the introduction of the different reactivity effects and reactivity coefficients, it is important to recognize that, apart from the limitations of the procedure that have already been outlined, there are other significant implications that should not be overlooked.

For example, suppose that the only reactivity effect of interest is due to density changes of the moderator. In addition assume that this density varies at the n^{th} power of the temperature of the moderator at each position of the reactor. If the neutron distribution in the reactor is independent of the moderator temperature, then the over-all reactivity effect calculated by using Eq. (1-12) will depend on the n^{th} power of some appropriate average temperature of the moderator. If, however, the neutron distribution is affected by the moderator temperature, which might be the case during severe transients, then the over-all reactivity will depend on some average moderator temperature raised to a power different from n . This example is important because it implies that what is calculated or measured microscopically may not necessarily apply directly to a particular severe reactor transient or to a particular reactor type.

Another important implication of the computationally expedient separation of reactivity into different effects is that sometimes some of these effects and the corresponding coefficients cannot be measured individually. For example, consider a heterogeneous reactor which experiences reactivity changes due to the Doppler effect and to fuel expansion. Both these effects depend on the temperature of the fuel. Any reactor experiment that affects the fuel temperature will disclose both effects at once. Only special experimental arrangements might allow the measurement of each individual effect but then these arrangements do not correspond to the reactor in question. Similar comments can be made about other reactivity effects and reactivity coefficients.

Finally, a few remarks are necessary regarding the omission or inclusion of the term $\Lambda \partial \ln(\Lambda F) / \partial t$ in the definition of reactivity. It is clear that this term depends primarily on the rate of change of the shape function $N_0(\vec{r}, E, \Omega, t)$, see Eq. (1-14). If the shape changes are rapid as a function of time, then this term is important. If shape changes are slow then this term may be neglected.

In attempting to approximate the effects of the rate of change of the shape function on reactivity by means of thermodynamic variables, some caution must be exercised with regard to what constitutes a proper approximation. To be specific, suppose that the macroscopic cross sections vary proportionately to some power k of a representative reactor temperature $\theta_m(t)$, and that the shape function varies in proportion to some other power ℓ of the same temperature. Then the reactivity effect experienced by the reactor is

$$\rho_1 = \rho - \Lambda d \ln(\Lambda F) / dt \simeq a \theta_m^{k+\ell}(t) + b \theta_m^{\ell-1}(t) [d \theta_m(t) / dt], \quad (1-28)$$

where $a, b = \text{constants}$. The meaning of Eq. (1-28) is that the dependence of the rate effects on the thermodynamic variables is different than the dependence of cross section effects. This result is different than the suggestion made in the literature [10] according to which, if ρ is approximated by, say, $a \theta_m(t)$, then the rate effect should be taken as $b d \theta_m(t) / dt$.

2 ANALYTICAL TECHNIQUES USED IN REACTOR SAFETY STUDIES

2.1 General Remarks

The gross dynamic behavior of a nuclear power plant may be represented by a set of ordinary differential equations, such as Eqs. (1-19) and (1-20), in conjunction with the equations that relate the variations of the quantities ρ , Λ and β_i , to the changes of the state of the reactor or, ultimately, to the energy stored in the reactor.

As already pointed out, in attempting to use these equations for analytical studies, serious questions of interpretation and computability of various quantities arise. For example, one question of interpretation is whether $P(t)$, ρ , Λ and β_i can be assigned specific physical meanings. Regardless of the interpretation that is adopted, these quantities will be referred to in this section by their accepted names (power, reactivity, etc.) keeping in mind the comments of Sec. 1.

Concerning the question of computability, it is evident that the computation of the integrals defining ρ , Λ and β_i during a severe transient encounters major difficulties because, in general, it is not possible to know the exact form of the integrands.

Disregarding these questions, the analytical problem of reactor safety may be approached from a purely mathematical point of view, that is, as the solution of a set of differential or integrodifferential equations of the form given by Eqs. (1-19) and (1-20). If in so doing general requirements can be derived which:

- when satisfied, guarantee the safe operation of the plant,
- can be tested by means of simple nonhazardous experiments, and
- are insensitive to small perturbations of the physical constants of the system,

then the questions of interpretation and computability lose their importance. It is the purpose of this section to show that this is indeed sometimes possible and to present a number of specifications that a large variety of safe reactors should satisfy.

In order to present some general mathematical techniques that are, or can be, used in the conceptual solution of the kinetics equations, in the following discussion it is assumed that the ratios β_i / Λ are physical invariants of each reactor and that reactivity can be expressed as a function or functional of the power or the stored energy.* The discussion is also limited to dynamic problems

*In order to improve the approximation implied by the assumed constancy of β_i / Λ the term

occurring over periods of time of the order of minutes or less.

The assumption about the constancy of $\bar{\beta}_1/\Lambda$ does not affect the qualitative aspects of the results that will be derived.** Also, the assumption about the dependence of reactivity on power is theoretically justifiable and it is consistent with the majority of the existing experimental evidence.

2.2 Linear Version of Reactor Kinetics

In the context of the previous comments, one simple form that the kinetics equations may take is

$$\frac{dP}{dt} = \frac{\rho - \bar{\beta}}{\Lambda} P + \sum_i^m \lambda_i C_i + Q \quad (2-1)$$

$$\frac{dC_i}{dt} = \frac{\bar{\beta}_i}{\Lambda} P - \lambda_i C_i, \quad i = 1, 2, \dots, m \quad (2-2)$$

$$\rho = \rho_e + \int_0^t f(t-\tau)[P(\tau) - P_0]d\tau, \quad (2-3)$$

where P_0 is a reference steady-state power level, ρ_e is the externally introduced reactivity, applied during $t \geq 0$, and $f(t)$ is a kernel determined, in general, from the various coefficients of reactivity and the time constants associated with each coefficient.*** The kernel $f(t) = 0$ for $t < 0$.

The integral in Eq. (2-3) accounts for reactivity introduced by the various reactivity effects which are experienced whenever the reactor changes its state. The implication of this integral is that all causes of reactivity change, such as temperatures for example, are related to power by linear differential equations with constant coefficients and that they are related to reactivity effects by means of constant coefficients of reactivity. This linear interdependence between all processes contributing to reactivity is an approximation and it is appropriate for small changes of power.

Equation (2-1) is nonlinear. The explicit solution of the system of Eqs. (2-1) through (2-3) is a formidable, if not impossible task. Specific solutions can be found, however, by considering limited regimes of reactivity variation.****

$d \ln(\Lambda F)/dt$ may be retained in the equation for $P(t)$, so that reactivity is $\rho_1 = \rho - \Lambda d \ln(\Lambda F)/dt$, but omitted from the delayed neutron precursor equations.

**The reason for this is that, when the solutions of the kinetics equations are bounded, the reactor system belongs to the class of systems that are classified mathematically as "structurally stable" with regard to the delayed neutron precursors. In other words, under this condition, variations of the values of $\bar{\beta}_1/\Lambda$ do not affect the qualitative properties of the solutions of the reactor equations.

***If the reactor is controlled by an external control system, the kernel $f(t)$ may also include the characteristics of the controller.

2.2.1 Slow Startup

As an example, consider the process of reactor startup. During startup the variation of the energy stored in the reactor system is very small and feedback effects are negligible, provided that the reactor is brought to power over a time interval of a few minutes. Thus, if the externally introduced reactivity is applied stepwise at $t=0$, the reactivity experienced by the reactor may be approximated by

$$\rho = \rho_0 + \rho_-, \quad \rho_0 = 0 \quad (t < 0), \quad (2-4)$$

where ρ_0 is the step reactivity and ρ_- is the reactivity when the reactor is shutdown. The reactor equations under these conditions reduce to the approximate form:

$$\frac{dP}{dt} = \frac{\rho_0 + \rho_- - \bar{\beta}}{\Lambda} P + \sum_i^m \lambda_i C_i + Q_0 \quad (2-5)$$

$$\frac{dC_i}{dt} = \frac{\bar{\beta}_i}{\Lambda} P - \lambda_i C_i, \quad i = 1, 2, \dots, m, \quad (2-6)$$

where Q_0 is a constant source. If before startup the reactor is at equilibrium, the balance equations are:

$$0 = \frac{\rho_- - \bar{\beta}}{\Lambda} P_0 + \sum_i^m \lambda_i C_{i0} + Q_0 \quad (2-7)$$

$$0 = \frac{\bar{\beta}_i}{\Lambda} P_0 - \lambda_i C_{i0}, \quad i = 1, 2, \dots, m, \quad (2-8)$$

where P_0 and C_{i0} are the steady state values. The solution of the set of Eqs. (2-5) and (2-6) can be easily found in terms of Laplace transforms [11]:

$$s\bar{P} - P_0 = \frac{\rho_0 + \rho_- - \bar{\beta}}{\Lambda} \bar{P} + \sum_i^m \lambda_i \bar{C}_i + \bar{Q}_0 \quad (2-9)$$

$$s\bar{C}_i - C_{i0} = \frac{\bar{\beta}_i}{\Lambda} \bar{P} - \lambda_i \bar{C}_i, \quad i = 1, 2, \dots, m. \quad (2-10)$$

If \bar{C}_i is eliminated from Eqs. (2-9) and (2-10) and the steady state conditions (2-7) and (2-8) are used, then it is found that:

$$\bar{P} = \frac{P_0 \left(\Lambda + \sum_i^m \frac{\bar{\beta}_i}{s + \lambda_i} \right) + \Lambda \bar{Q}_0}{\Lambda s + \sum_i^m \frac{\bar{\beta}_i s}{s + \lambda_i} - (\rho_0 + \rho_-)} \quad (2-11)$$

****Some representative digital computer codes for numerical solution of the space-independent kinetics equations, Eqs. (2-1) and (2-2), are described in Appendix I of this chapter. Appendix I was prepared by Harold Greenspan of Argonne National Laboratory.

The inverse transform of \bar{P} gives the time-dependent function $P(t)$. The general features of $P(t)$ depend on the roots of the denominator of \bar{P} . This denominator is related to the well-known inhour equation. Indeed, when the source level is zero ($Q_0 = \rho_- = 0$), the reactor is initially critical and the inhour equation for step reactivity changes determines the roots of the denominator of \bar{P} :

$$\rho_0 = \Lambda s + \sum_{i=1}^m \frac{\bar{\beta}_i s}{s + \lambda_i} \quad (2-12)$$

For $\rho_0 > 0$, Eq. (2-12) admits one positive root and m negative roots. The negative roots correspond to exponentially decaying terms in $P(t)$. The positive root corresponds to an exponentially rising term in $P(t)$. The inverse of the positive root is usually referred to as the asymptotic period. When $\rho_0 < 0$, all roots are negative and the critical reactor is shutdown. For details about the inverse transform of \bar{P} and the experimental interpretation of the inhour equation, the reader is referred to the literature [12, 13].

It is important to realize that the assumption that the feedback effects are negligible is meaningful only when the step changes in reactivity are of the order of cents and the approximation is used only over a limited period of time. Also it must be realized that, in the presence of a noisy source term different from zero, the problem of calculating the power behavior is one of probabilistic estimation rather than of a deterministic computation [14] (see also Criticality chapter).

The zero feedback approximation is useful because it provides the operator of a reactor with a good insight into the temporal behavior of the reactor during a slow startup and it constitutes a simple technique for the calibration of the reactivity worth of different reactor components.

2.2.2 Small Perturbations of Reactivity

Another example of limited reactivity variation is the case when the externally introduced reactivity is perturbed in such a way that the changes of all state variables of the reactor are sufficiently small that nonlinear effects can be neglected. Thus if before the reactivity perturbation is introduced, the reactor is at a critical or operating steady state and if, after the perturbation, each of the state variables is written as the sum of the steady-state value plus an increment, then Eqs. (2-1) through (2-3) reduce to the form:

$$\frac{dp}{dt} = \frac{P_0}{\Lambda} \rho_e - \frac{\bar{\beta}}{\Lambda} p + \frac{P_0}{\Lambda} \int_0^t f(t-\tau) p(\tau) d\tau + \sum_{i=1}^m \lambda_i c_i, \quad Q = 0 \quad (2-13)$$

$$\frac{dc_i}{dt} = \frac{\bar{\beta}_i}{\Lambda} p - \lambda_i c_i, \quad i = 1, 2, \dots, m, \quad (2-14)$$

where the lower case letters denote the increments with respect to the steady-state values and where second or higher order terms have been neglected. Note that Eqs. (2-13) and (2-14) constitute the linear version or the first approximation of reactor kinetics equations of any form. In other words, if the variations of all the state variables are small enough, then the dynamics of any reactor can be represented by the set of Eqs. (2-13) and (2-14). Of course, to each reactor and each particular operating point there will correspond a different kernel $f(t)$.

Considering ρ_e as the input and p as the output of the reactor system, the solution of Eqs. (2-13) and (2-14) in terms of Laplace transforms is:

$$\frac{\bar{p}}{\rho_e} = \frac{P_0 R(s)}{1 - P_0 R(s) F(s)} = H(s), \quad (2-15)$$

where

$$R(s) = \frac{1}{\Lambda s + \sum_{i=1}^m \frac{\bar{\beta}_i s}{s + \lambda_i}} \quad (2-16)$$

is the normalized zero power reactor transfer function and $F(s)$ is the Laplace transform of $f(t)$ [11, 13]. Equation (2-15) is shown in Fig. 2-1 by means of block diagrams.

The inverse transform of \bar{p} can be written as:

$$p(t) = \int_0^t h(t-\tau) \rho_e(\tau) d\tau, \quad (2-17)$$

where $h(t)$ is the inverse transform of $H(s)$ or the reactor system impulse response. The transform $H(s)$ is the reactor transfer function at power. The meaning of Eqs. (2-15) or (2-17) is that, for the types of reactivity variations specified above, the time-dependent power increment can be explicitly calculated provided that the reactor transfer function $H(s)$ or the equivalent impulse response $h(t)$ is known.

The general features of $p(t)$ for a given well-behaved $\rho_e(t)$ depend on the poles of the transfer function $H(s)$, namely, on the location of the roots of the characteristic equation:

$$1 - P_0 R(s) F(s) = 0. \quad (2-18)$$

These poles are often also referred to as the eigenvalues of the first approximation. Note that, in the

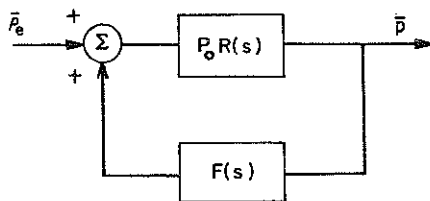


FIG. 2-1 Block diagram of Eq. (2-15).

present case, they depend on the value of the steady-state power, P_0 .

If the poles are in the left half of the complex plane, the solution $p(t)$ is asymptotically stable and after a long time reaches the value

$$p(t)_{t \rightarrow \infty} = \left[s \bar{p}_e \frac{P_0 R(s)}{1 - P_0 R(s) F(s)} \right]_{s \rightarrow 0} \quad (2-19)$$

A necessary but not sufficient condition for the existence of a physically meaningful limit is that:

$$F(0) = \int_0^{\infty} f(t) dt < 0. \quad (2-20)$$

In terms of reactivity coefficients and associated time constants, the meaning of inequality (2-20) is that the sum of the weighted coefficients of reactivity must be negative. The weighting factors are the time constants of, and the fractions of the energy stored in, the various regions of the reactor that are responsible for contributions to reactivity.

To see this argument clearly, consider the linearized version of the kinetics equations written explicitly in terms of all the state variables that are pertinent to the calculation of the feedback reactivity. These equations can always be written in the canonical form (see also Sec. 1.5):

$$\frac{dp}{dt} = \frac{P_0}{\Lambda} \rho_e - \frac{\beta}{\Lambda} p + \frac{P_0}{\Lambda} \sum_j r_j \theta_j + \sum_i \lambda_i c_i \quad (2-21)$$

$$\frac{dc_i}{dt} = \frac{\beta_i}{\Lambda} p - \lambda_i c_i, \quad i = 1, 2, \dots, m \quad (2-22)$$

$$\frac{d\theta_j}{dt} = a_j p - g_j \theta_j, \quad j = 1, 2, \dots, n, \quad (2-23)$$

where θ_j are the increments of the canonical thermodynamic variables and r_j the coefficients of reactivity corresponding to these variables. The system of Eqs. (2-21) through (2-23) is completely equivalent to Eqs. (2-13) and (2-14). In particular:

$$f(t) = \sum_j r_j a_j e^{-g_j t} \quad (2-24)$$

$$F(0) = \int_0^{\infty} f(t) dt = \sum_j r_j (a_j / g_j) \quad (2-25)$$

Equation (2-25) proves the statement that inequality (2-20) requires the sum of the weighted coefficients of reactivity to be negative with the weighting factors being the time constants, $1/g_j$, and the measures, a_j , of the fractions of the total energy stored in the various regions, j .

This important requirement is also necessary even when the feedback reactivity is not a linear functional of power, provided that to each critical state of the reactor there corresponds one and only

one combination of values of the state variables. In other words, suppose that a reactor is critical and then it is excited by an external reactivity, ρ_e . If there is one and only one set of changes of the reactor variables which compensate for ρ_e and lead to another critical state, then it is necessary that inequality (2-20) be satisfied.

If the poles of the transfer function $H(s)$ are in the right half of the complex plane, the solution $p(t)$ of the linearized equations is unstable. This instability, however, is true both mathematically and physically only for small changes of the reactor variables. It must be recognized that the ultimate behavior of the power is governed by the nonlinearities that have been neglected; when the power is growing the assumed conditions for linearization are no longer valid. Nonlinear effects will be discussed later.

The problem of finding the roots of Eq. (2-18), namely, the problem of investigating the stability of linear systems, is discussed in many textbooks [15, 16]. The techniques that have been developed for this purpose are Bode diagrams, Nyquist plots, the root locus, etc. All these techniques lead to identical results cast in different forms, and the choice among them is a matter of convenience or personal preference. These techniques will not be discussed in detail here. It suffices only to note that in the case of a nuclear reactor the poles are a function of the steady state power P_0 and therefore, even though the reactor may be linearly stable for a range of values of P_0 , it may become linearly unstable for values of P_0 outside this range.

In view of the approximate nature and limited range of applicability of Eqs. (2-13) and (2-14), the justifiable question is often raised about the real importance of the linearized or transfer function approach to the problem of reactor dynamics analysis, or, stated differently, about the connection, if any, between the exact solution and the one derived from the linearized model.

Several remarks are appropriate with regard to this question. First, the solution of the linearized equations, either in the frequency domain [Eq. (2-15)] or in the time domain [Eq. (2-17)], suggests definite and easily implementable experimental procedures which permit the determination of $H(s)$ or the equivalent $h(t)$, as well as $R(s)$ and $F(s)$. These procedures are discussed later.

Second, from a purely mathematical standpoint (as discussed in many textbooks), much information can be gained from the linear approximation about the nonlinear solution. For example, if one requires asymptotic stability, then the solution of the first approximation must necessarily be asymptotically stable. Also, if one is interested in the solutions for small perturbations, the linear solutions in many cases yield an adequate approximation. Of course, the magnitude of perturbations for which the linear equations are valid, even though mathematically well defined, [17, 18] is often difficult to evaluate quantitatively in practice because of lack of information about the exact form of the nonlinearities or the complexity of the problem. As a rule of thumb, for nuclear reactors, variations of the order of tenths of the steady-state values of the state variables may generally be considered as adequately represented by linear

approximations. Beyond that level the nonlinearities become important.

2.3 Nonlinear Reactor Kinetics—Welton's Sufficient Criterion of Stability

Consider again the set of Eqs. (2-1) through (2-3) in which feedback reactivity is a linear functional of power. As already stated, no general analytical solution for these equations has been derived. Specific results have been found only by means of computer codes or analog simulators. Even though there is no general analytical solution, techniques are available for the investigation of the stability of the solutions.

For example, Welton [19, 20] has derived a sufficient criterion which guarantees the asymptotic stability of the reactor power to all bounded variations of external reactivity for all steady state power levels P_0 . This criterion states that, if

$$\operatorname{Re} F(j\omega) = \int_0^{\infty} f(t) \cos \omega t dt \leq 0, \quad (2-26)$$

then the reactor is asymptotically stable. In other words, if the real part of the Laplace transform of the feedback kernel along the $j\omega$ -axis is non-positive, then this is sufficient to assure asymptotic stability of the reactor power with respect to all bounded variations of external reactivity. The meaning of the criterion is that the phase of the negative feedback transfer function $[-F(s)]$ along the $j\omega$ -axis is between -90° and $+90^\circ$. In simple practical terms this implies that the linearized version of the dynamic equations of the reactor without delayed neutrons should admit stable solutions at all power levels P_0 and that the negative feedback transfer function belongs to the restricted class of positive real functions or input impedance type of functions of passive electric networks.

Welton's criterion is very appealing because it relates nonlinear stability to properties that characterize the linear behavior of the reactor. Unfortunately, though, this sufficient criterion is over-restrictive for several reasons.

First, the reactor can be linearly stable without delayed neutrons even if the negative feedback transfer function is not a positive real function.

Second, the sufficient criterion is non-constructive even for simple reactor systems since both theoretical considerations and a variety of experimental results indicate that most reactors do become linearly unstable after the steady state power reaches a particular level. In view of the fact that the criterion is only sufficient, there is no a priori reason to believe that reactors which are not linearly stable at all power levels are necessarily nonlinearly unstable. As a matter of fact, there is a lot of experimental evidence to the contrary.

Third, it is not realistic to require that the reactor be stable at all possible power levels since this can never be achieved in practice due to constraints imposed by the reactor materials. It is, therefore, important to examine whether, from the analytical standpoint, it is possible to es-

tablish less restrictive requirements which are applicable to a larger variety of reactors than Welton's criterion.

To this end many authors [21-23] have considered Liapunov's direct method as an alternate approach to the problem of reactor stability. It turns out that in most of the specific reactor problems that have been treated in the literature by Liapunov's direct method, the derived requirements for asymptotic stability are at best equivalent to, if not more restrictive than, Welton's criterion. This fact has not been generally recognized. It is not the purpose of the present chapter to go into a detailed proof of the preceding statement. Briefly, it may be said that Liapunov's method has been applied to reactor dynamics by neglecting delayed neutrons and transforming the remaining equations into a canonical form similar to that proposed by Lur'e and Letov [24]. Thus, sufficient conditions for stability are derived by means of a variety of Liapunov functions of the same type used by Lur'e and discussed by Popov [25] and others. It can be shown [26] that the sufficient requirements thus derived are at best equivalent to Welton's.

It is felt that the restrictive results derived so far by Liapunov's method do not necessarily represent an inherent limitation of the method but rather a shortcoming of the procedures that were used for its implementation. Liapunov's method is both unique and powerful and should be further pursued for the analysis of questions of nonlinear reactor dynamics.

Whether Liapunov's method or any other technique is used for analysis of reactor stability, there are a number of important practical aspects that must be incorporated in the analysis in order to arrive at useful results. These are discussed in the following sections.

2.4 Some Practical Considerations of Nonlinear Reactor Kinetics

2.4.1 General Remarks

From the discussion in the preceding section, it follows that analytical results on nonlinear reactor stability are not satisfactory. It is the purpose of this section to discuss a number of changes which might be introduced in the approach to the problem of analysis of reactor stability in order to arrive at some practical results.

2.4.2 The Practical Importance of Delayed Neutrons

In most analyses of reactor dynamics, the delayed neutrons are neglected from the kinetics equations. In other words, for analysis purposes, it is assumed that $\beta_1 = \lambda_1 = 0$. The motivation for this omission is the fact that the complexity of the equations is greatly reduced. The justification for the omission is that if a reactor is linearly stable without delayed neutrons, it is even more stable when the delayed neutron precursors are taken into account. This fact has been proved several times in the literature. In addition, it can be also rigorously shown that in the presence of feedback effects, if a reactor can be proven to be nonlinearly

stable without delayed neutrons, then it is more stable when delayed neutrons are included in the analysis [27]. Mechanically speaking, the delayed neutrons always add extra dumping into the system.

From the practical standpoint, however, the omission of delayed neutrons is a gross oversimplification. First, it turns out that analytical results derived without delayed neutrons may be so conservative that they are impractical. To appreciate this fact, suppose that a reactor becomes linearly unstable after a certain critical steady-state power level P_{cr} . Assume that this critical level is established through analysis of the reactor model without delayed neutrons and that its value is A . Next, suppose that the same analysis is carried out with the delayed neutron precursor equations included in the linearized kinetics equations. It can be readily shown that, in general, the value of the critical power level is $B > A$. In fact, in reactors with relatively long heat transmission time constants (of the order of a fraction of a second and above) and prompt lifetimes smaller than 10^{-4} sec, the difference between B and A may be one order of magnitude or more [9]. Needless to emphasize, this is such a substantial difference that the conservative estimate becomes impractical.

Another reason that renders the omission of delayed neutrons impractical is that any experimental verification of theoretical results will necessarily include the effects of delayed neutrons.

2.4.3 The Admissible Operating Power Levels

Many analyses of nonlinear reactor dynamics attempt to establish requirements for stability at all possible operating power levels $P_0 (0 < P_0 < \infty)$. In other words, these analyses deal with the problem of stability to unrestricted initial perturbations or the problem of global stability.

Designing, however, a reactor to be globally stable or globally asymptotically stable at all operating power levels is an unrealistic and unnecessary target to aim for. It is unnecessary because environmental constraints require that the operating power level P_0 does not exceed some specified and limited value, compatible with the materials and heat transmission processes partaking in the operation of the reactor. It is unrealistic because reactors, already built and having an excellent safety record, do become even linearly unstable and yet not destructively so, because of inherent safeguarding nonlinearities.

Similar comments can be made about the energy stored in the various regions of a reactor. Without repeating the arguments, it is obvious that it is not practical to require that a reactor be capable of storing unlimited amounts of energy.

2.4.4 The Feedback Reactivity

Apart from the preceding considerations with regard to delayed neutrons, operating power level and energy storage, there is another aspect of the model for feedback reactivity that has been used so far, which renders results on nonlinear stability questionable. Specifically, it is the practice of assuming that feedback reactivity is linearly related to power, see Eq. (2-3). The implication

of this assumption is that the various reactivity effects remain decoupled no matter how large the deviations from equilibrium are and, furthermore, that these effects are always linearly related to the stimuli that initiate them. In addition, Eq. (2-3) implies that every reactor is capable of providing an unlimited feedback reactivity of either sign.

It is clear that all these interrelated implications of the linear model for feedback reactivity are not consistent with the concept of reactivity, as a review of the assumptions behind the formal definition of reactivity immediately reveals (see Sec. 1). In particular, the feedback reactivity cannot grow indefinitely with power because of the limited number of neutrons per fission.

2.4.5 Asymptotic Versus Lagrangian Stability

Quite often, analysis of nonlinear reactor dynamics attempts to establish requirements that are sufficient to guarantee asymptotic stability of the reactor power and of all the other variables with respect to all, or a large variety of, perturbations from equilibrium. These requirements may often be very restrictive, if not impossible for the reactor designer to meet. In addition, even if the reactor is designed to be asymptotically stable, in practice the inherent reactor noise leads to a mode of operation characterized by state variables that are bounded but not asymptotically stable. It is, therefore, of interest to examine other types of stability.

One possibility is to attempt to establish sufficient requirements which guarantee Lagrangian stability rather than asymptotic stability [18]. Specifically, it may be expedient to design a safe reactor so that all state variables remain bounded within certain predetermined upper and lower bounds without concern about their exact temporal behavior within these bounds. Geometrically speaking, designing for Lagrangian stability implies that the designer is satisfied with the assurance that all the reactor variables are confined within a closed region of the multi-dimensional state variable space instead of requiring that they tend to coalesce at one point.

It turns out that the conditions for Lagrangian stability may be more relaxed with respect to the conditions for asymptotic stability [28]. This is to be expected since, so to speak, it is intuitively understandable that it is easier to reach one or all of many points than to reach a single point.

Lagrangian stability is neither better nor worse than asymptotic stability. In certain cases asymptotic stability may be essential to the operation of a system as, say, when it is desired to keep the temperature of a room at exactly 70°F. In other cases, however, a given margin may be just as tolerable. It is felt that this is the case of reactor systems and it is suggested that this fact be incorporated in the design procedures pertaining to stability.

2.5 A Practical Model for Nonlinear Reactor Stability and Some of Its Properties

The purpose of this section is to indicate how consideration of some of the comments of Sec. 2.4

might improve the requirements for nonlinear reactor stability and lead to practical results.

As indicated in Sec. 2.4.4 the physical nature of the fission process is such that the feedback reactivity is an ultimately bounded function with respect to any and all the state variables of the reactor.

Given an equilibrium state of the reactor, feedback reactivity changes around this state might be expressed as:

$$\rho = \rho_e + \rho_f, \quad \rho_f = z + \phi(\theta_1, \dots, \theta_n),$$

$$z = \sum_j^n r_j \theta_j, \quad (2-27)$$

where ρ_e is the externally introduced reactivity, ρ_f is the feedback reactivity, θ_j are the different perturbations of state variables which may introduce a reactivity change, r_j are the coefficients of reactivity when the latter is represented by a linear approximation and $\phi(\theta_1, \dots, \theta_n)$ is a nonlinear function of $(\theta_1, \dots, \theta_n)$ such that the feedback reactivity ρ_f is bounded both for positive and negative values of z . The exact values of the coefficients r_j and the exact analytical expression of $\phi(\theta_1, \dots, \theta_n)$ depend on the reactor type and the prevailing operating conditions. Regardless of the exact form of $\phi(\theta_1, \dots, \theta_n)$, however, it is evident that it must satisfy the inequalities:

$$0 < \phi(\theta_1, \dots, \theta_n) < |z| \text{ for } z < -z_1;$$

$$\phi(\theta_1, \dots, \theta_n) < 0, \quad |\phi(\theta_1, \dots, \theta_n)| < z \text{ for } z > z_2, \quad (2-28)$$

where z_1, z_2 are given positive quantities and $\phi(0) = 0$.

The complete set of reactor dynamics equations that must be considered in the light of the comments of the preceding sections is:

$$\frac{dP}{dt} = \frac{\bar{\rho} - \bar{\beta}}{\Lambda} P + \sum_i^m \lambda_i C_i, \quad (2-29)$$

$$\frac{dC_i}{dt} = \frac{\bar{\beta}_i}{\Lambda} P - \lambda_i C_i, \quad i = 1, 2, \dots, m, \quad (2-30)$$

$$\frac{d\theta_j}{dt} = a_j(P - P_0) - g_j \theta_j, \quad j = 1, 2, \dots, n, \quad (2-31)$$

$$\rho = \rho_e + z + \phi(\theta_1, \dots, \theta_n), \quad z = \sum_j^n r_j \theta_j. \quad (2-32)$$

The coefficients a_j, g_j may be taken as constant or variable but bounded. The basic difference between Eqs. (2-1) through (2-3) and Eqs. (2-29) through (2-32) is the introduction of the nonlinear bounded variation of feedback reactivity and the admission of coefficients a_j, g_j that may be variable but bounded.

The set of Eqs. (2-29) through (2-32) has some useful properties that are independent of the exact analytical form of $\phi(\theta_1, \dots, \theta_n)$ and/or a_j, g_j . Before proceeding with the enumeration of these properties, it is helpful to reduce the equations to a more convenient form.

To this end, consider the cases where the externally introduced reactivity is a step, $\rho_e = \text{constant}$, applied at time $t = 0$.^{*} In addition, for algebraic simplicity assume that $P_0 = 0$. Under these conditions suppose that the reactor admits only two equilibrium states. One state is the shutdown state, ($P = C_i = \theta_j = 0$), and the other is the state that the reactor will attempt to reach at the end of the transient initiated by the reactivity ρ_e , ($P_\infty, C_{i\infty}, \theta_{j\infty}, \rho = 0$). The latter is also called the operating equilibrium state and can be determined from the solution of the algebraic system of equations:

$$\rho_e + z_\infty + \phi(\theta_{1\infty}, \dots, \theta_{n\infty}), \quad z_\infty = \sum_j^n r_j \theta_{j\infty}; \quad (2-33)$$

$$\frac{\bar{\beta}_i}{\Lambda} P_\infty - \lambda_i C_{i\infty} = 0, \quad i = 1, 2, \dots, m; \quad (2-34)$$

$$a_{j\infty} P_\infty - g_{j\infty} \theta_{j\infty} = 0, \quad j = 1, 2, \dots, n, \quad (2-35)$$

where the coefficients $a_{j\infty}, g_{j\infty}$, if variable, are evaluated at $(P_\infty, C_{i\infty}, \theta_{j\infty})$. Mathematically, the system of algebraic Eqs. (2-33) through (2-35) may admit more than one solution. This possibility, however, is excluded from the present discussion. In fact, when the feedback reactivity is a monotonic function of z , the necessary condition for the existence of one and only one operating equilibrium state is

$$\sum_j^n \frac{r_j a_{j\infty}}{g_{j\infty}} < 0 \quad (2-36)$$

for all practical operating conditions, as simple inspection of Eqs. (2-33) through (2-35) immediately reveals. This condition is identical with the condition for the existence of the solution of the linear approximation that was derived in Sec. 2.2.2.

If all the variables are measured with respect to the operating equilibrium state and each variable is normalized with respect to its own equilibrium value ($\neq 0$), then the system of Eqs. (2-29) through (2-32) reduces to the form:

$$\frac{dp}{dt} = -\frac{\bar{\beta}}{\Lambda} p + \sum_i^m \frac{\bar{\beta}_i}{\Lambda} c_i + \sum_j^n \frac{\gamma_j}{\Lambda} \theta_j + R(p, \theta); \quad (2-37)$$

^{*}The results that will be derived can readily be generalized to cases where ρ_e is variable but uniformly bounded with respect to time [29].

$$\frac{dc_i}{dt} = \lambda_i(p - c_i), \quad i = 1, 2, \dots, m; \quad (2-38)$$

$$\frac{d\theta_j}{dt} = g_{j\infty}(p - \theta_j) + R_j(p, \theta_j), \quad i, j = 1, 2, \dots, n, \quad (2-39)$$

where the lower case letters p, c_i denote the dimensionless normalized increments with respect to the operating equilibrium values ($P_\infty, C_{i\infty}$), θ_j denotes again the dimensionless normalized increment of θ_j , $\gamma_j = r_{j\infty} a_{j\infty} P_\infty / g_{j\infty}$ and

$$R(p, \theta_j) = \frac{1}{\Lambda} \sum_{j=1}^n \gamma_j \theta_j p + \frac{\phi((1 + \theta_1), \dots, (1 + \theta_n)) - \phi(1, \dots, 1)}{\Lambda} (1 + p)$$

$R_j(p, \theta_j) = 0$ for $a_j, g_j = \text{constant}$ or

$$= g_{j\infty} \left(\frac{a_j}{a_{j\infty}} - 1 \right) (p + 1) \text{ for } a_j, g_j = \text{variable}. \quad (2-40)$$

In this normalized formulation, the operating state is ($p = c_i = \theta_j = 0$) while the shutdown state is ($p = c_i = \theta_j = -1$). In addition, the stability of the reactor may now be viewed as the stability of the equilibrium states with respect to arbitrary initial conditions of the state variables rather than step perturbations of the external reactivity. Also note that the assumed boundedness of the feedback reactivity and the coefficients a_j, g_j implies that the functions $R(p, \theta_j)$ and $R_j(p, \theta_j)$ vary at most linearly with p and θ_j ($i = 1, 2, \dots, n$) for large deviations from equilibrium.

It is a simple matter to prove that, for the cases of practical interest, the shutdown state is unstable, as it should be, otherwise it would not be possible to start up the reactor.

Regarding the operating state, the following properties are inherent in the form of Eqs. (2-37) through (2-39) and independent of the exact functional dependence of ρ_f, a_j, g_j .

a. Either all state variables are constant or all are unbounded. The assumed uniqueness of the operating equilibrium state implies that regardless of what the initial conditions are for the set of Eqs. (2-37) through (2-39), no solutions exist which consist of some constant and some diverging state variables as $t \rightarrow \infty$.

Indeed, suppose that only $\theta_j = M$ as $t \rightarrow \infty$. Thus $\frac{d\theta_j}{dt} \Big|_{t \rightarrow \infty} = 0$ and consequently p would also tend

to a constant as $t \rightarrow \infty$ [Eq. (2-31)]. But if p tends to a constant then all c_i [Eq. (2-30)] and θ_j [Eq. (2-31)] would also tend to a constant and therefore the limiting values of all the variables would be given by the set of algebraic Eqs. (2-33) through (2-35) which is assumed to admit only one solution.

b. No state variable admits a finite escape time. The form of Eqs. (2-37) through (2-39) excludes the possibility of any of the state variables increasing indefinitely over a finite period of time. In other words, there are no finite escape times.

Indeed, since for large values of p and θ_j the nonlinear functions $R(p, \theta_j)$ and $R_j(p, \theta_j)$ vary at most linearly with p and all θ_j , it is always possible to write that, for

$$K = p^2 + \sum_{i=1}^m c_i^2 + \sum_{j=1}^n \theta_j^2 \geq H_0,$$

$$H_0 = \text{positive constant}, \quad (2-41)$$

all the rates of change of the state variables are such that

$$\frac{dp}{dt} \text{ or } \frac{dc_i}{dt} \text{ or } \frac{d\theta_j}{dt} \leq F(p, c_i, \theta_j) K^{1/2}, \quad (2-42)$$

where $F(p, c_i, \theta_j)$ is some continuous positive and bounded function with respect to all its arguments. Consequently,

$$\begin{aligned} \frac{d}{dt} K &= 2p \frac{dp}{dt} + 2 \sum_{i=1}^m c_i \frac{dc_i}{dt} + 2 \sum_{j=1}^n \theta_j \frac{d\theta_j}{dt} \\ &\leq 2F(p, c_i, \theta_j) K, \quad K(t > t_0) \geq H_0, \end{aligned} \quad (2-43)$$

or

$$\int_{t_0}^T \frac{dK}{K} \leq 2 \int_{t_0}^T F(p, c_i, \theta_j) dt. \quad (2-44)$$

Note that if some state variable admits a finite escape time T , the left hand side of inequality (2-44) will tend to infinity while the right hand side will be finite since the integrand $F(p, c_i, \theta_j)$ is a bounded function. Clearly this is absurd and no finite escape times exist.

The practical importance of this result is that growing instabilities, if any, take a relatively "long time" to reach high levels and therefore they might be controllable. In order to be able to qualify quantitatively the meaning of "long time" it is necessary to have additional information about the function ρ_f and the coefficients a_j, g_j . In this connection, the following property is informative.

c. Existence of continuously growing instabilities. Having established that there are no finite escape times, it is permissible to approximate the feedback reactivity by some suitable linear functional of power, or equivalently by a linear combination of the state variables θ_j , provided that the approximation is used over a finite time only. Note that there is no inconsistency here with regard to the discussion of Sec. 2.4.4 because the linear approximation is taken only over a finite time.

Thus, the dynamic behavior of the reactor is

described now by Eqs. (2-37) through (2-39) with $\phi(\theta_1, \dots, \theta_n) = 0$ and suitable average definitions of \bar{a}_j , \bar{g}_j , $\bar{r}_j = \text{constant}$, or by Eqs. (2-1) through (2-3), where

$$\bar{f}(t) = \sum_j^n \bar{r}_j \bar{a}_j e^{\bar{g}_j t}. \quad (2-45)$$

In order for this representation to be meaningful, Eq. (2-36) must be satisfied. In other words:

$$\int_0^\infty \bar{f}(t) dt = \sum_j^n \bar{r}_j \bar{a}_j / \bar{g}_j < 0, \quad (2-46)$$

so that the approximate system does admit one and only one operating equilibrium state as the original system does.

In addition, suppose that

$$\bar{f}(t) < 0 \text{ for all } t > 0, \quad (2-47)$$

i.e., the kernel of the average linear approximation is negative.

Under these conditions no solution of the reactor equations exists which contains a continuously increasing power level. Indeed, if the opposite were true and the power were always increasing and positive, there would exist some time T such that the reactivity seen by the reactor,

$$\int_0^T \bar{f}(t - \tau) p(\tau) d\tau, \quad (2-48)$$

would become negative and increasing in absolute value. Then, the power equation indicates that $p(t)$ should be decreasing, a result that contradicts the assumed continuous growth of $p(t)$. Consequently, no solution exists with a continuously increasing power level.

The practical meaning of this result is that, if a reactor is designed so that the linear approximation around all possible operating power levels is such that

$$f(t) < 0 \text{ and } \int_0^\infty f(t) dt < 0, \quad (2-49)$$

then growing instabilities, if any, will be experienced in the form of diverging oscillations. Since the reactor power is physically bounded for negative values ($p(t) > -1$), the positive half-cycle of any oscillation lasts a shorter time than the corresponding negative half-cycle* and, therefore, again, diverging oscillations, if any, take a relatively long time to the extent that corrective action may be taken.

*If this were not true, under the assumed conditions, the feedback reactivity would have a constant sign and thus no oscillation could exist.

Of course, there is an important question here which has not been answered. Specifically, "Do diverging oscillations exist?" or, to put it differently, "What are the conditions that prevent the existence of diverging oscillations?" This question cannot be answered without further knowledge of the functional dependence of feedback reactivity on the state variables. Some special examples have been treated in the literature (see also chapter on Fast Transients).

Even though the existence or nonexistence of diverging oscillations cannot be discussed in general terms, it is felt that the proposed practical model for reactor kinetics and its properties have yielded some useful results. Specifically, the nonexistence of finite escape times, the nonexistence of continuously growing instabilities and the difference in time length of the positive and negative half-cycles of oscillations all tend to provide the designer with the confidence that, if the premises on which these conclusions are based, namely inequalities (2-49), are satisfied, then it may be possible to keep likely accidents under control and to avoid undesirable consequences. The results are also very useful for space-time dynamic studies. They provide the analyst with some general properties which must be satisfied by any approximate space-time representation that may be proposed. For example, suppose that the transport theory equations are approximated and solved in space and time by means of a computer. If the computer solutions indicate a finite escape time, then the analyst knows that he has introduced the wrong approximation.

2.6 A Desirable Model for Feedback Reactivity that Guarantees Practical Safety

The purpose of this section is to derive a general requirement on the behavior of feedback reactivity which guarantees the practical safety of the reactor. Practical safety means a self-limiting transient performance that entails a tolerable amount of energy storage in the reactor with regard to a given maximum external reactivity input.

To this end, given an operating power level P_0 , reduce the kinetics equations into one equation of the form:

$$\frac{dp(t)}{dt} = f_0(p(t))(p(t) + P_0) - \int_0^t d(t - \tau) p(\tau) d\tau, \quad (2-50)$$

$$d(t) = \sum_i^m \frac{\bar{\beta}_{i0}}{\Lambda_0} (\delta(t) - \lambda_i e^{-\lambda_i t}) \text{ for } t > 0, \\ = 0 \text{ for } t < 0, \quad (2-51)$$

where $\bar{\beta}_{i0}/\Lambda_0$ are the asymptotic values of $\bar{\beta}_i/\Lambda$ around the operating level P_0 , $\delta(t)$ is the delta function and $f_0(p(t))$ is a functional of the incremental power $p(t)$ ($f_0(0) = 0$) that accounts for the feedback reactivity and the variations of Λ , $\bar{\beta}_i$ and the normalization factor F (Sec. 1). Eq. (2-50) is quite

general and represents the dynamics of any reactor with respect to arbitrary initial perturbations.

Next multiply both sides of Eq. (2-50) by $p(t)$ and integrate the result with respect to time to find:

$$\begin{aligned} \frac{1}{2} p^2(t) + \int_0^t \int_0^\tau d(\tau - \lambda) p(\tau) p(\lambda) d\tau d\lambda \\ - \int_0^t p(\tau) f_0(p(\tau)) [p(\tau) + P_0] d\tau = \frac{1}{2} p^2(0). \end{aligned} \quad (2-52)$$

The second term in the left hand side of Eq. (2-52) is positive definite, regardless of the values of $p(t)$. This can be proved either formally by using the procedure proposed by Bochner [30] or heuristically by observing that $d(t)$ may be interpreted as the input impedance of a RC passive electric network and the double integral as the total energy supplied to this network.

It is evident from Eq. (2-52) that, if there exists a time $t = T_1$ beyond which the third integral in the left hand side of this equation becomes and remains negative, then $p(t)$ must be decreasing. In addition, if there exists a time $t = T_2 > T_1$ such that the energy stored in the reactor during that time is tolerable, if $p(t)$ has reached a level comparable to the rate of withdrawal of energy from the reactor and/or possibly, but not necessarily, if the time interval T_2 is adequate for external controls to be initiated, then the reactor is for all practical purposes safe.

It is recognized that the above general specifications on the behavior of reactivity do not indicate the explicit dependence of feedback reactivity on the stored energy in the reactor and its distribution throughout the reactor. It is felt, however, that these specifications in conjunction with Eq. (2-52) provide a simple and general way for verifying whether different feedback reactivity models are capable of guaranteeing practical safety. The following example illustrates the point.

A possible model for the functional $f_0(p(\tau))$ is

$$f_0(p(\tau)) = \int_0^\tau f(t - \tau) p(\tau) d\tau + f_1(p(\tau)), \quad (2-53)$$

where the functional $f_1(p(\tau))$ is such that $f_0(p(\tau))$ is bounded both for positive and negative values. After some elementary algebra, Eq. (2-51) can be written as:

$$\begin{aligned} \frac{d}{dt} \left[\int_0^t \int_0^\tau k(\tau - \lambda) p(\tau) p(\lambda) d\tau d\lambda \right] \\ = p^2(t) \int_0^t f(t - \tau) p(\tau) d\tau + p(t) [p(t) + P_0] f_1(p(\tau)), \end{aligned} \quad (2-54)$$

where $k(t)$ is the inverse Laplace transform of the function:

$$K(s) = s + D(s) - P_0 F(s) \quad (2-55)$$

and $D(s)$ and $F(s)$ the Laplace transforms of $d(t)$ and $f(t)$, respectively. Note that $K(s)$ is the inverse of the reactor transfer function at power P_0 ($K(s) = 1/H(s)$).

If it is assumed that

$$\int_0^\infty f(t) dt < 0, \quad f(t) < 0,$$

$$K(s) \text{ a positive real function,} \quad (2-56)$$

then the reactor power is ultimately bounded. Indeed, when $K(s)$ is a positive real function, the double integral in the left hand side of Eq. (2-54) is a positive definite function. Its time derivative becomes negative for large positive values of $p(t)$.

If $f(t) < 0$ and $\int_0^\infty f(t) dt < 0$ and the power is positive

and increasing when $p(t) > p_1$, then $\int_0^t f(t - \tau) p(\tau) d\tau < 0$ and increasing in absolute value. At this level $f_1(p(\tau)) > 0$ and is varying slower than linearly with $\int_0^t f(t - \tau) p(\tau) d\tau$ since $f_0(p(\tau))$ must be a bounded function. Thus, the sign of the right hand side of Eq. (2-54) is determined by the first term and it is negative. Since $p(t)$ is physically bounded for $p(t) < -1$, it is concluded that $p(t)$ is ultimately bounded outside some range $(-1, p_1)$ or, in other words, that there are no diverging oscillations.

In addition, the positive real character of $K(s)$ (or $H(s)$) guarantees that $p(t)$ is asymptotically stable with respect to small perturbations of the operating power level P_0 .

The above conclusions are based on satisfying conditions (2-56) which can be easily implemented either theoretically or experimentally. It must be noted, however, that asymptotic stability with regard to small perturbations and ultimate boundedness do not exclude the possibility of bounded oscillations.

2.7 Space-Dependent Reactor Kinetics

The analysis of reactor stability through investigation of the properties of the space-independent kinetics equations is necessary in order to establish whether the reactor variables have a tendency to grow beyond tolerable levels or not.

As already pointed out, however, conclusions derived from the space-independent kinetics equations pertain only to the time-dependent growth factors of the state variables. These growth factors may be viewed as average attributes of power, temperatures, etc., prevailing in a reactor. Undoubtedly when these average attributes tend to grow beyond tolerable limits the reactor is unsafe. On the other hand, if they remain bounded or if they have an asymptotic value, this does not necessarily mean that the reactor is safe. The reason is that even though, on the average, the reactor variables may be well behaved, local extrema may occur which lead to undesirable consequences. For example, suppose that the power in a bundle of fuel

elements behaves like a self-limiting burst which is a desirable state of affairs. Assume, however, that at the same time the energy stored in these elements is so high that before it has a chance to be transmitted to the coolant it melts the fuel. Obviously this may have serious consequences.

The implications of these comments are that guarantees for stability derivable from the space-independent reactor kinetics equations are not the only requirements that must be met by a reactor design. What is mostly needed, after stability in the above sense is guaranteed, is an analysis of space-dependent kinetics or in other words derivation of complete solutions of space- and time-dependent equations. Not much analytical work has been done in this area for the simple reason that the problem is very difficult. Some computer codes have been developed for boiling water reactors [7], and a flux synthesis technique has been proposed [6]. (See also Sec. 5.3 of Chapter 9.)

It is interesting to note here that in attempting to study space-dependent kinetics the concept of reactivity must be abandoned or generalized. The problem must be approached by devising new space- and time-dependent approximations to the basic transport theory balance equations. There is no a priori reason why these new approximations must be cast in terms of the concept of reactivity.

3 MEASUREMENT OF LINEAR DYNAMIC CHARACTERISTICS OF NUCLEAR REACTOR SYSTEMS

3.1 General Remarks

The discussion of Sec. 2 indicates that when the variations of the output power of a reactor system are relatively small, then the time behavior of these variations can be calculated from a knowledge of the input reactivity and the impulse response, $h(t)$, or its Laplace transform, the transfer function $H(s)$. Similar relations can be derived for any other pair of variables viewed as input and output.

When the variations of the reactor state variables are large, then the computation of the exact time behavior of these variables is a much more difficult problem. However, under certain conditions, knowledge of the properties of the linear approximation which corresponds to small variations may yield useful information about large variations.

In view of these remarks it is useful to have experimental techniques which allow the study of the various impulse responses or transfer functions of reactor systems so that analytical results can be verified experimentally.

The purpose of this section is to discuss some of the techniques that can be used for the measurement of linear dynamic characteristics either in the time or in the frequency domain. Such techniques are oscillation, crosscorrelation and autocorrelation tests. In addition, some consideration is given to the problem of measurement of nonlinear characteristics or experimental identification of nonlinear systems. As already mentioned, this is a much more involved problem and by no means fully understood yet (see also chapter on

Mathematical Models of Fast Transients).

3.2 Oscillation Tests

In order to establish the theoretical foundation of oscillation tests, consider the relationship between power and reactivity when both undergo relatively small variations. In the frequency domain this relationship is [Eq. (2-15)]:

$$\bar{p}(s) = H(s)\bar{\rho}_e(s). \quad (3-1)$$

The partial fraction expansion [11] of $\bar{p}(s)$ contains two types of terms. The first depend on the poles of the input $\bar{\rho}_e(s)$ and the second depend on the poles of the transfer function $H(s)$. Specifically:

$$\bar{p}(s) = \sum_i^q \frac{k_i}{s - s_i} + \sum_j^r \frac{k_j}{s - w_j}, \quad (3-2)$$

where k_i , k_j are the residues, s_i are the poles of the input ($i = 1, 2, \dots, q$) and w_j are the poles of the transfer function ($j = 1, 2, \dots, r$). For simplicity and without loss of generality, all poles have been assumed single so that the residues are derived from the simple relations:

$$k_i = (s - s_i)\bar{p}(s)|_{s=s_i} \quad \text{or} \quad k_j = (s - w_j)\bar{p}(s)|_{s=w_j}. \quad (3-3)$$

The inverse Laplace transform of $\bar{p}(s)$ is:

$$p(t) = \underbrace{\sum_i^q k_i e^{s_i t}}_{\text{steady-state response}} + \underbrace{\sum_j^r k_j e^{w_j t}}_{\text{transient response}}. \quad (3-4)$$

If all the poles, w_j , of the transfer function are in the left half complex plane, then after a sufficiently long time all the terms that correspond to these poles, namely the transient response, will become practically zero. After that time, $p(t)$ reaches its steady-state response which is characterized only by the poles of the input and the corresponding residues.

The steady-state response, when the input reactivity varies sinusoidally, is:

$$\rho_e = A \sin \omega t, \quad \bar{\rho}_e(s) = A \frac{\omega}{s^2 + \omega^2} \quad (3-5)$$

$$\text{steady-state } p(t) = k_1 e^{j\omega t} + k_2 e^{-j\omega t}, \quad (3-6)$$

where $k_2 = \bar{k}_1 = \text{conjugate of } k_1$,

$$k_1 = \left[(s - j\omega) A \frac{\omega}{s^2 + \omega^2} H(s) \right]_{s=j\omega} = \frac{A}{2j} H(j\omega). \quad (3-7)$$

Therefore:

$$\text{steady-state } p(t) = A |H(j\omega)| \sin(\omega t + \angle H(j\omega)). \quad (3-8)$$

The meaning of Eq. (3-8) is that the steady-state response of a reactor, or any other linear system, excited by a sinusoidal input is also a sinusoid of the same frequency. Its relative amplitude and phase with regard to the input are given by the magnitude and phase of the transfer function evaluated at the same frequency as that of the input, respectively.

Equation (3-8) suggests a simple steady-state experiment for the measurement of the transfer function along the frequency axis $s = j\omega$. Indeed, if the reactor is excited by means of sinusoidally varying reactivities of different frequencies, the relative amplitudes and phases of the corresponding steady-state power oscillations yield the magnitude and phase of the transfer function at these frequencies. The experimental results may be plotted either in polar form or as Bode diagrams or in any other convenient diagrams [15].

Knowledge of the values of the transfer function $H(s)$ for $s = j\omega$ is sufficient to determine $H(s)$ for all values of s because this function is analytic everywhere except at its poles and $H(j\omega)$ can be readily analytically continued for all s [31]. In addition, by taking the inverse transform of $H(j\omega)$ it is possible to determine the impulse response, $h(t)$, if it is so desired.

This is the essence of oscillation tests which are steady state experiments resulting in the measurement of linear dynamic characteristics. It is evident that the technique can be implemented to measure the transfer function between any pair of reactor variables one of which is considered as the input and varied sinusoidally and the other as the output.

For details of the experimental setup for the measurement of transfer functions by means of oscillation tests, the reader is referred to the abundant literature on the subject [32, 33].

Transfer function measurements through oscillation tests have been used for the measurement of prompt neutron lifetimes, coefficients of reactivity and associated time constants, cross sections, the prediction of power level at which linear stability is lost, shutdown reactivity, etc.

Before closing this brief discussion on oscillation tests, it is worth indicating some of the limitations that arise when the method is applied to a "real reactor system." The power of a presumably exactly critical reactor is not constant and the steady-state response to a sinusoidal input reactivity is not a true sinusoid. The reason for the deviations from the ideal performance is that practically all physical phenomena are to some degree statistical in nature or subject to small perturbations due to unpredictable environmental changes. For example, the fission process and/or the boiling process vary statistically in a reactor or the criticality may be influenced by atmospheric temperature changes, etc.

The net effect of these inherent or externally stimulated statistical fluctuations on oscillation tests performed on reactors is that either the input reactivity amplitude may be required to be large enough so that the power oscillations have an amplitude much larger than the statistical fluctuations, or the power oscillations must be Fourier-analyzed in order to extract the fundamental frequency com-

ponent and reduce the contribution from the inherent fluctuations.

Increasing the amplitude of the input reactivity is undesirable on two counts. First, the output is distorted because the reactor does not behave like a linear system. Of course, it is possible to extract the fundamental component of the output by Fourier analysis but then the end result is a measurement of the describing function rather than the transfer function [33]. Second, it may be unsafe or mechanically difficult to use large amplitude reactivity inputs.

Fourier analysis of the power oscillations is in a sense a crosscorrelation of the output with a sinusoid. Consequently, if crosscorrelation is to be used in order to reduce the errors due to statistical fluctuations, it is not necessary to excite the reactor by a pure sinusoidal input reactivity. Any small amplitude periodic reactivity waveform will accomplish the same purpose. The reason is that any periodic waveform (square wave, saw-tooth, etc.) may be visualized as a sum of pure sinusoids. Each sinusoid results in its own contribution to the steady-state response of the output power. Thus by comparing the fundamental component of the output with the fundamental component of the input, it is possible to measure the transfer function.

Two other implications of the presence of statistical fluctuations in reactor systems are: (a) the need for Fourier analysis of the power oscillations suggests that it may be more appropriate to excite the reactor with an input reactivity that contains a broad band of frequencies with equal amplitude and to crosscorrelate this input with the corresponding output. The result of this operation is again the measurement of the dynamic characteristic between the input and the output (the impulse response) as discussed in Sec. 3.3; (b) the statistical fluctuations contain information concerning the dynamics of the reactor. Under certain conditions, this information may be extracted by autocorrelating the power or any other observable variable fluctuations, as discussed in Sec. 3.4.

3.3 Crosscorrelation Tests

The basis of crosscorrelation tests is the convolution integral relationship between input and output [Eq. (2-14)]:

$$p(t) = \int_0^t h(t-\tau) \rho_e(\tau) d\tau = \int_0^t h(\tau) \rho_e(t-\tau) d\tau. \quad (3-9)$$

If the autocorrelation of a function, $x(t)$, is defined as

$$\phi_{xx}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau) dt, \quad \phi_{xx}(\tau) = \phi_{xx}(-\tau) \quad (3-10)$$

and the crosscorrelation between two functions, $x(t)$ and $y(t)$, is defined as

$$\phi_{xy}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau) dt, \quad \phi_{xy}(\tau) = \phi_{yx}(-\tau), \quad (3-11)$$

where T is the period of the functions $x(t)$ and $y(t)$, when these functions are periodic, or $T \rightarrow \infty$ when the functions are stochastic or aperiodic, then the crosscorrelation of $\rho_e(t)$ and $p(t)$ is

$$\begin{aligned}\phi_{\rho_e p}(\tau) &= \frac{1}{T} \int_{-T/2}^{T/2} \rho_e(t - \tau) p(t) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} dt \rho_e(t - \tau) \int_0^t d\lambda h(\lambda) \rho_e(t - \lambda) \\ &= \int_0^t d\lambda h(\lambda) \frac{1}{T} \int_{-T/2}^{T/2} dt \rho_e(t - \tau) \rho_e(t - \lambda) \\ &= \int_0^t h(\lambda) \phi_{\rho_e \rho_e}(\tau - \lambda) d\lambda. \quad (3-12)\end{aligned}$$

The meaning of Eq. (3-12) is that the autocorrelation function of the input $\rho_e(t)$ is related to the crosscorrelation of $\rho_e(t)$ and $p(t)$ by the same convolution integral as $\rho_e(t)$ is related to $p(t)$.

If the input reactivity $\rho_e(t)$ is a broad band signal,* its autocorrelation function is approximately equal to a delta function:

$$\phi_{\rho_e \rho_e}(t) = A^2 \delta(t), \quad A^2 = \text{constant}, \quad (3-13)$$

and Eq. (3-12) yields

$$\phi_{\rho_e p}(\tau) = A^2 h(\tau). \quad (3-14)$$

In other words, the crosscorrelation of the input and the output is proportional to the impulse response when the input is a broad band signal. Eq. (3-12) can also be written in the frequency domain. To this end, use must be made of two-sided Laplace transforms [15] because correlation functions are defined both for positive and negative time shifts τ . Thus,

$$\bar{\phi}_{\rho_e p}(s) = H(s) \bar{\phi}_{\rho_e \rho_e}(s) \quad (3-15)$$

or

$$\bar{\phi}_{\rho_e p}(j\omega) = H(j\omega) \bar{\phi}_{\rho_e \rho_e}(j\omega). \quad (3-16)$$

In other words, the transfer function for $s = j\omega$ is equal to the ratio of the power spectrum of the crosscorrelation of the input and the output over the power spectrum of the autocorrelation of the input.

Crosscorrelation tests are essentially an experimental implementation of either Eq. (3-14) or (3-16). For the details of the experimental procedure, the reader is referred to the literature [34, 35]. For the purposes of this discussion, it suffices to note that the presence of inherent statistical fluctuations is not as restrictive as in oscillation

tests [36]. The reason is that the properties of the broad band input are in general completely independent of the statistics of the fluctuations. In fact, through proper choice of the input waveform, the results of crosscorrelation tests may also be unaffected by the nonlinear characteristics even for large input amplitudes. This topic is discussed in Sec. 3.6.

3.4 Autocorrelation Tests

The use of autocorrelation tests, to extract information about the linear dynamic characteristics of a reactor, entails a series of assumptions about the cause of the fluctuations and the statistical attributes of this cause.

To make ideas specific, assume that the power fluctuations $p(t)$ or those of any other variable, around a constant mean value, are due to inherent and statistically predictable variations of reactivity, $\rho_e(t)$. Thus, the relationship between $p(t)$ and $\rho_e(t)$ is given by Eq. (3-9). The autocorrelation function of $p(t)$ is

$$\begin{aligned}\phi_{pp}(\tau) &= \frac{1}{T} \int_{-T/2}^{T/2} p(t) p(t + \tau) dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} dt \int_0^t d\lambda h(\lambda) \rho_e(t - \lambda) \int_0^{t+\tau} d\mu h(\mu) \rho_e(t + \tau - \mu) \\ &= \int_0^t d\lambda h(\lambda) \int_0^{t+\tau} d\mu h(\mu) \frac{1}{T} \int_{-T/2}^{T/2} dt \rho_e(t - \lambda) \rho_e(t + \tau - \mu) \\ &= \int_0^t d\lambda \int_0^{t+\tau} d\mu h(\lambda) h(\mu) \phi_{\rho_e \rho_e}(\tau + \lambda - \mu). \quad (3-17)\end{aligned}$$

Eq. (3-17) can best be understood in the frequency domain. Its two-sided Laplace transform is

$$\bar{\phi}_{pp}(s) = H(s) H(-s) \bar{\phi}_{\rho_e \rho_e}(s), \quad (3-18)$$

and for $s = j\omega$

$$|H(j\omega)|^2 = \frac{\bar{\phi}_{pp}(j\omega)}{\bar{\phi}_{\rho_e \rho_e}(j\omega)}. \quad (3-19)$$

The meaning of Eq. (3-19) is that the square of the amplitude of the transfer function $H(s)$ for $s = j\omega$ is equal to the ratio of the power spectrum of the autocorrelation of the output over the power spectrum of the autocorrelation of the input. Under the assumption that $\rho_e(t)$ is an inherent statistical perturbation, the power spectrum of $\phi_{\rho_e \rho_e}(\tau)$ cannot

be readily measured. If, however, this spectrum is assumed as flat and equal to a constant over the frequency range of interest, then the measurable spectrum of the fluctuations of $p(t)$ is proportional to the square of the magnitude of the transfer function. In other words, the magnitude of the transfer function can be measured by simply autocorrelating the power fluctuations and finding the power spec-

*The bandwidth of the signal must be equal or broader than the bandwidth of the transfer function.

trum.

Note that autocorrelation tests do not disclose any information about the phase of $H(j\omega)$. The phase can be computed from the magnitude only when the transfer function belongs to the class of phase minimum functions such as the impedances or admittances of linear, passive, lumped parameter electrical networks [15, 37].

The great appeal of autocorrelation tests is that information about the linear dynamic characteristics of the reactor can be derived without externally perturbing the reactor but from the fluctuations that are superimposed on the steady state record of any reactor variable. A major drawback of the method is that it is not always possible to verify whether the power spectrum of the autocorrelation of the presumed statistical excitation is indeed flat white. Thus, resonance tendencies of $\bar{\phi}_{\rho e \rho e}(j\omega)$ might be attributed to the transfer function and erroneous conclusions be derived.

3.5 A Stability Monitor

When the linear or nonlinear dynamic behavior of a reactor system is adequately characterized by the linear characteristics, as in some of the examples presented in Sec. 2, then the crosscorrelation method can be used to continuously monitor the quality of the transient and stability properties of the reactor [38].

To this end, the reactor is continuously perturbed by a small amplitude, broad band reactivity input which is then crosscorrelated on line with the resulting output. The crosscorrelation is performed simultaneously for a large number of time shifts τ . The outputs of the crosscorrelators are depicted on a fluorescent screen and provide a graph of the impulse response, $h(t)$. This graph is readily interpretable to predict any poor transient response or instability tendencies that may be developing in the reactor.

3.6 Representation and Identification of Nonlinear Systems

The purpose of this section is to present some of the techniques that are currently pursued for the analytical representation and experimental identification of nonlinear, lumped parameter, stationary and physically realizable systems which from an engineering standpoint can be visualized in terms of input-output data. These techniques have not yet been fully utilized in the nuclear field but they are suggestive of possible developments and they provide a theoretical basis for studies of nonlinearities.

The discussion is first presented without reference to any particular physical system, type of input or output. Only one input and one output are considered for mathematical expediency. The formalism can be easily generalized to any number of inputs and/or outputs.

Some of the results are used to indicate the effects of nonlinearities in measurements performed in reactor systems.

3.6.1 The Functional Representation of Nonlinear Systems

Suppose that a nonlinear system is excited by an input $x(t)$ which results in an output $y(t)$. A possible interpretation of this visualization is that the input $x(t)$ enters the system, it is processed by the system and then appears as an output $y(t)$. Therefore, measurements of the input and the output contain all the information about the dynamics of the system. For the types of systems under consideration, this information can be stated analytically in terms of a functional,

$$y(t) = F(x(t - \tau)); \quad \tau < t. \quad (3-20)$$

The physical meaning of the analytical statement (3-20) is that the present value of the output is uniquely determined only by the past history of the input. The exact form of the functional, of course, depends on the specific system on which the input-output measurements are taken.

The functional F can always be expanded into an infinite series of functionals of the input, a form that is more suggestive of possible approaches to the problems of representation and identification of nonlinear systems. Specifically, it can easily be shown that it is always possible to write [39]:

$$\begin{aligned} y(t) &= F(x(t - \tau)) \\ &= h_0 + \int_0^\infty h(t - \tau)x(\tau)d\tau \\ &\quad + \int_0^\infty \int_0^\infty h(\tau_1, \tau_2)x(t - \tau_1)x(t - \tau_2)d\tau_1d\tau_2 + \\ &\quad \dots + \int_0^\infty \dots \int_0^\infty h_k(\tau_1, \dots, \tau_k)x(t - \tau_1) \\ &\quad \dots x(t - \tau_k)d\tau_1 \dots d\tau_k + \dots \end{aligned} \quad (3-21)$$

In principle, the nonlinear system is now characterized by an infinite set of kernels $h_k(\tau_1, \dots, \tau_k)$. The contribution from each kernel to the output is derived from a generalized convolution operation. Equation (3-21) is called a Volterra expansion. Note that, if the system is linear, $(h_k(\tau_1, \dots, \tau_k) = 0, k \geq 2)$, the functional expansion reduces to the well-known linear convolution plus the constant h_0 .

Functionals of the type appearing in Eq. (3-21) have been studied by Volterra [40]. Wiener has used the functional expansion to investigate nonlinear electrical network problems [41]. Other authors have investigated different properties of functionals and developed a systematic algebra and multi-Laplace transformation theory for a system or combination of systems represented by functional expansions [42-44]. The findings of these authors will not be discussed here. The reader is referred to reference [45] for an excellent summary.

For the purposes of this presentation, it suffices to emphasize that if the kernels $h_k(\tau_1, \dots, \tau_k)$ were known, then the nonlinear system would be completely represented; in other words, if the kernels

could be measured, then the nonlinear system would be completely identified.

Equation (3-21), as it stands now, is not always convenient for experimental interpretation. However, it is suggestive of a similar expansion which is more suitable both for theoretical and experimental studies. Specifically, if the expansion were in terms of an orthogonal set of functionals, a number of advantages are evident:

If the expansion is truncated at a finite number of functionals, the input is approximated in the least mean square error sense.

Each member of the expansion is linearly independent of the others and the number of functionals considered.

Each member of the expansion can be determined by use of the orthogonality relationship which is, in essence, a generalized crosscorrelation procedure.

The advantages of an expansion in terms of orthogonal functionals can be achieved when the input is a gaussian white noise, as rigorously proved by Wiener [41] and Chesler [42]. Wiener's theory is briefly summarized in the next section.

3.6.2 Wiener's Canonical Representation of Nonlinear Systems

Wiener's rigorous theory of representation of nonlinear systems is described in his monograph "Nonlinear Problems in Random Theory" [41]. Only some important results are repeated here for convenience.

Consider a gaussian white noise signal $x(t)$ defined over all times from $-\infty$ to ∞ and having a power spectrum equal to unity. Given an arbitrary, symmetrical kernel $K_n(\tau_1, \tau_2, \dots, \tau_n)$ and the functional

$$f_n(t) = \int_0^\infty \dots \int_0^\infty K_n(\tau_1, \dots, \tau_n) x(t - \tau_1) \dots x(t - \tau_n) d\tau_1 \dots d\tau_n, \quad (3-22)$$

it can readily be shown that the average value of this functional over all times is

$$\begin{aligned} \frac{1}{2T} \int_{-T}^T f_n(t) dt &= 0 \quad \text{for } n = \text{odd} \\ &= (2m-1)(2m-3) \dots (1) \int_0^\infty \dots \int_0^\infty K_n(\tau_1, \tau_1, \dots, \tau_m, \tau_m) d\tau_1 \dots d\tau_m \\ &\quad \text{for } n = 2m = \text{even}. \quad (3-23) \end{aligned}$$

On the basis of Eq. (3-23) it is easy to prove that the following G_n functionals are orthogonal regardless of the values of the symmetric kernels $K_n(\tau_1, \dots, \tau_n)$.

$$G_0 = \text{constant}$$

$$G_1(K_1, x, t) = \int_0^\infty K_1(\tau) x(t - \tau) d\tau$$

$$G_2(K_2, x, t) = \int_0^\infty \int_0^\infty K_2(\tau_1, \tau_2) x(t - \tau_1) x(t - \tau_2) d\tau_1 d\tau_2 - \int_0^\infty K_2(\tau, \tau) d\tau$$

$$G_3(K_3, x, t) = \int_0^\infty \int_0^\infty \int_0^\infty K_3(\tau_1, \tau_2, \tau_3) \times x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) d\tau_1 d\tau_2 d\tau_3 - 3 \int_0^\infty \int_0^\infty K_3(\tau_1, \tau_1, \tau) x(t - \tau) d\tau_1 d\tau$$

$$G_n(K_n, x, t) = \sum_{\nu=0}^{[n/2]} a_{n-2\nu}^{(n)} \int_0^\infty \dots \int_0^\infty K_n(\tau_1, \dots, \tau_n) x(t - \tau_1) \dots x(t - \tau_{n-2\nu}) \cdot \delta(\tau_{n-2\nu+1} - \tau_{n-2\nu+2}) \dots \delta(\tau_{n-1} - \tau_n) d\tau_1 \dots d\tau_n$$

where $[n/2] = n/2$ for $n = \text{even}$ and

$[n/2] = (n-1)/2$ for $n = \text{odd}$ and

$$a_{n-2\nu}^{(n)} = (-1)^\nu \frac{n!}{2^\nu (n-2\nu)! \nu!}.$$

In other words, the functional G_1 is orthogonal to all constants, the functional G_2 is orthogonal to all constants and all functionals G_1 , etc. The orthogonality is defined over the time variable t :

$$\frac{1}{2T} \int_{-T}^T G_n(K_n, x, t) G_m(K_m, x, t) dt = \delta_{nm} \quad \text{for } n \neq m \quad (3-24a)$$

$$\begin{aligned} & \frac{1}{2T} \int_{T \rightarrow -\infty}^{\infty} G_n(K_n, x, t) G_n(L_n, x, t) \\ &= n! \int_0^{\infty} \dots \int_0^{\infty} K_n(\tau_1, \dots, \tau_n) L_n(\tau_1, \dots, \tau_n) d\tau_1, \dots, d\tau_n, \end{aligned} \quad (3-24b)$$

where $L_n(\tau_1, \dots, \tau_n)$ is also a symmetric kernel.

The orthogonal functionals G_n form a complete set. Therefore, any function $y(t)$ which is square integrable can be expanded in a series of G_n functionals in a unique way:

$$y(t) = \lim_{N \rightarrow \infty} [G_0 + G_1(K_1, x, t) + \dots + G_N(K_N, x, t)]. \quad (3-25)$$

If the gaussian, white noise signal $x(t)$ is the input of a nonlinear system and $y(t)$ is the corresponding output, then Eq. (3-25) constitutes Wiener's canonical representation of the system and the kernels $K_n(\tau_1, \dots, \tau_n)$ are the characteristic kernels of the system. These kernels are related to the kernels $h_n(\tau_1, \dots, \tau_n)$ of the Volterra expansion, Eq. (3-21). To illustrate the point, suppose that the Volterra expansion has only two terms,

$$\begin{aligned} y(t) &= \int_0^{\infty} h_1(\tau) x(t - \tau) d\tau + \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_3(\tau_1, \tau_2, \tau_3) \\ &\quad x(t - \tau_1) x(t - \tau_2) x(t - \tau_3) d\tau_1 d\tau_2 d\tau_3 \end{aligned} \quad (3-26)$$

then the Wiener representation has also two terms and the kernels are related by

$$\begin{aligned} K_1(t) &= h_1(t) + 3 \int_0^{\infty} h_3(\tau_1, \tau_1, t) d\tau_1, \\ K_3(t_1, t_2, t_3) &= h_3(t_1, t_2, t_3). \end{aligned} \quad (3-27)$$

Therefore, knowledge of either set of kernels is adequate to characterize the nonlinear system.

The important difference between the two representations is that Wiener's orthogonal expansion suggests a simple experimental procedure for the measurement of the kernels $K_n(\tau_1, \dots, \tau_n)$ while the same is not true for the Volterra kernels $h_n(\tau_1, \dots, \tau_n)$. This procedure is discussed in the next section.

3.6.3 Measurement of the Wiener Kernels

The measurement of the kernels $K_n(\tau_1, \dots, \tau_n)$ is effectively an experimental implementation of the orthogonality relationships [Eq. (3-24)]. Indeed, suppose that the gaussian white noise signal $x(t)$ is fed simultaneously into the physical system under

investigation and another known system consisting of n pure delays in parallel. All the outputs from the pure delays are multiplied together and the output of the multiplier is also multiplied by the output $y(t)$ of the physical system and then integrated over a long time. The net result of this integration is proportional to the n^{th} Wiener kernel $K_n(\tau'_1, \dots, \tau'_n)$, where τ'_i are the delay times ($\tau'_1 \neq \tau'_2 \neq \dots \neq \tau'_n$). Indeed, the n delays and the first multiplier may be thought of as a known system with a Wiener representation:

$$\begin{aligned} x(t - \tau'_1) \dots x(t - \tau'_n) &= \int_0^{\infty} \dots \int_0^{\infty} \delta(\tau_1 - \tau'_1) \dots \\ &\quad \dots \delta(\tau_n - \tau'_n) x(t - \tau_1) \dots x(t - \tau_n) d\tau_1, \dots, d\tau_n \\ &= \int_0^{\infty} \dots \int_0^{\infty} L_n(\tau_1, \dots, \tau_n) x(t - \tau_1) \dots \\ &\quad \dots x(t - \tau_n) d\tau_1, \dots, d\tau_n = G_n(L_n, x, t), \end{aligned} \quad (3-28)$$

where

$$L_n(\tau_1, \dots, \tau_n) = \delta(\tau_1 - \tau'_1) \delta(\tau_2 - \tau'_2) \dots \delta(\tau_n - \tau'_n). \quad (3-29)$$

In other words, the set of n delays and their multiplier can be represented by a special Wiener functional of n^{th} order. If the output of the physical system is visualized as a sum of Wiener functionals, $G_n(K_n, x, t)$, then $G_n(L_n, x, t)$ is orthogonal to all these functionals except $G_n(K_n, x, t)$. Therefore, the output of the integrator is [Eq. (3-24b)]:

$$\int_{-\infty}^{\infty} y(t) G_n(L_n, x, t) dt = n! K_n(\tau'_1, \tau'_2, \dots, \tau'_n). \quad (3-30)$$

This proves the statement that the output of the integrator is proportional to the n^{th} Wiener kernel. It is evident that by changing the values of the delays τ'_i , the entire range of values of $K_n(\tau'_1, \dots, \tau'_n)$ can be measured. Since the number of delays, n , can be chosen at will, all kernels can be measured.

The experimental procedure for the measurement of the Wiener kernels is a generalized cross-correlation procedure. The necessary cross-correlations can be performed on line or by means of digital and/or analog computers.

Wiener's representation and identification procedure for nonlinear systems is conceptually very simple. It also shows that, just as gaussian white noise is adequate for the complete characterization of linear systems, it is also adequate for the characterization of nonlinear systems.

The use of gaussian white noise, however, is a

major drawback because of the necessity of extremely long integration times for crosscorrelation. The long times are necessary because of the serious experimental errors that are otherwise introduced. This is the reason why the method has not yet found wide application.

In spite of its shortcomings Wiener's canonical representation is very useful for a variety of theoretical and practical investigations.

In particular, in any practical system the functional representation will necessarily be truncated after a small number of functionals. In addition the useful bandwidth of any physical system is limited. These two observations suggest that it may be possible to design periodic signals defined over finite intervals of time and such that their pseudostatistical properties are similar to those of gaussian white noise up to a certain degree. Thus it will be possible to use the Wiener representation and perform practical measurements. Work along these lines is currently under way with promising preliminary results [46].

It is worth noting that the proposed procedure for the design of test signals for nonlinear systems is identical to the procedure used for linear systems. For example, in linear systems instead of using a gaussian white noise signal, it is adequate to use a periodic signal which over a time interval of one period has an autocorrelation function approximately equal to a delta function. In nonlinear systems the approximate comparison between the periodic signal and gaussian white noise must be carried not only up to the first correlation but also to higher order correlations.

3.6.4 Comparison of Oscillation and Autocorrelation Tests Performed on Reactors in the Presence of Nonlinearities

When the variations of the input to a reactor system are small, the reactor behaves like a linear system and oscillation or autocorrelation tests may yield almost the same information, namely, the transfer function of the reactor. When the variations are large, however, the reactor behaves like a nonlinear system and the question arises as to whether these two types of tests yield the same information.

To answer this question, suppose that the reactor power variations are represented by means of a Wiener canonical expansion:

$$p(t) = \sum_0^{\infty} G_n(K_n, \rho_e, t). \quad (3-31)$$

When the reactivity is varied sinusoidally, contributions to the fundamental frequency component of $p(t)$ arise from all the odd-order kernels $G_{2m+1}(K_{2m+1}, \rho_e, t)$ only. Thus, Fourier analysis of the output yields the describing function which depends only on the kernels $K_1(\tau)$, $K_3(\tau_1, \tau_2, \tau_3)$, etc. [33]. On the other hand, if the reactivity is gaussian, white noise and the output fluctuations are autocorrelated, the result depends on all the kernels. Indeed, since the functionals G_n are orthogonal to each other, only functionals of the same order correlate with each other and

$$\bar{\phi}_{pp}(\tau) = \sum_n \frac{1}{2T} \int_{T-\infty}^{\infty} G_n(K_n, \rho_e, t) G_n(K_n, \rho_e, t + \tau) dt. \quad (3-32)$$

Consequently, the spectrum of this autocorrelation function is different from the square of the amplitude of the describing function derived by means of oscillation tests.

3.6.5 Use of the Describing Function for Stability Studies

The describing function has been used to derive stability criteria for nonlinear reactor dynamics [47, 48]. The representation of a nonlinear system by means of functional expansions is quite informative regarding the question of the practical value of such criteria.

The describing function depends only on the odd-order kernels. Consequently, stability criteria derived from the describing function are bound to be incorrect when the even-order kernels contribute appreciably to the output of the system.

In fact, stability criteria derived from functional expansions are in general more restrictive than necessary. The reason is that such criteria usually reflect the mathematical conditions for the existence of the particular functional expansion chosen rather than the conditions for the stability of the sum of the functionals.

To see this point clearly, consider the simple function

$$y(t) = \frac{1}{1+t}, \quad t > 0. \quad (3-33)$$

This function is a well behaved function for all values of $t > 0$. Suppose now that $y(t)$ is written in the form of a power series:

$$y(t) = 1 - t + t^2 - t^3. \quad (3-34)$$

The series exists and converges to $y(t)$ only when $|t| < 1$. The implication of this simple example is that if the convergence of $y(t)$ were derived from its power series expansion, the range of t would be unnecessarily limited, even though the actual function $y(t)$ is bounded and convergent for all values of $t > 0$. Since a functional expansion may be thought of as a generalized power series expansion, this example shows exactly the difficulty that renders functional expansions impractical for stability studies of reactor systems.

APPENDIX I REPRESENTATIVE DIGITAL COMPUTER CODES FOR SPACE-INDEPENDENT REACTOR KINETICS

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A number of digital computer codes have been written, and a variety of techniques [49] have been applied to obtain accurate and efficient numerical

solutions for the space-independent, one-energy group, reactor kinetics equation systems.

These equations have been derived in Sec. 2 of this chapter, see Eqs. (2-1) and (2-2). For convenience in discussing the computer codes two forms of the kinetic equations are repeated below:

$$\frac{dn}{dt} = (\rho' - 1)\beta \frac{n}{\Lambda} + \sum_{i=1}^I \lambda_i C_i + Q$$

$$\frac{dC_i}{dt} = \beta_i \frac{n}{\Lambda} - \lambda_i C_i, \quad (A-1)$$

and

$$\frac{dn}{dt} = [k_{eff}(1 - \beta) - 1] \frac{n}{\Lambda} + \sum_{i=1}^I \lambda_i C_i + Q$$

$$\frac{dC_i}{dt} = k_{eff}\beta_i \frac{n}{\Lambda} - \lambda_i C_i. \quad (A-2)$$

The nomenclature applicable to this Appendix are:

- n = neutron density.
- k_{eff} = effective multiplication constant.
- $k_{ex} = k_{eff} - 1$.
- β = delayed neutron fraction.
- $\rho = k_{ex}/k_{eff}$ = reactivity;
- $\rho' = (k_{ex}/k_{eff})/\beta$ = reactivity in dollars
- C_i = density of precursors for the i th delayed neutron group.
- β_i = delayed neutron fraction for the i th delayed neutron group.
- λ_i = decay constant of the i th delayed neutron group.
- Λ = prompt neutron lifetime.
- I = total number of delayed neutron groups.
- Q = external neutron source.

A zero subscript to the above quantities (where applicable) indicates an initial value.

The codes cited here (listed alphabetically) have been selected on the basis that they represent a method used, or they have some particularly useful feature like the reactivity-neutron density relation, etc., not available in other codes.

More extensive kinetic code listings and descriptions may be found in references [50] and [51].

A.1 AIREK II, AIREK III codes

The AIREK codes [52, 53] have been written in FORTRAN for the IBM-704, 709, and 7090. These programs solve the kinetic systems Eqs. (A-1) coupled with equations describing the reactivity ρ' either as a function of time or in terms of a feedback function F . The following relation is assumed for F :

$$\frac{dF}{dt} = a_1 F + a_2 n(t) + a_3 n(0), \quad (A-3)$$

where a_1 , a_2 , and a_3 are input parameters. The F equation with $a_3 = 0$ has the same form as the precursor equation; this fact is utilized in the sub-

sequent solution.

AIREK II solves the coupled system using a fourth-order Runge-Kutta method. AIREK III obtains a substantially improved numerical solution by using a modified system of equations and a fifth-order Runge-Kutta procedure. The modified system is derived by assuming $n(t) = \exp[J(t)]$ and dividing Eqs. (A-1) by $n(t)$.

In both programs subroutines are available for the following forms for ρ' :

$\rho' = \rho'_0$, ρ'_0 being a fixed step input reactivity;

$$\rho' = \rho'_0 + \sum_m D_m F_m,$$

D_m being constant and F_m as described in Eq.

(A-3);

$\rho' = \rho'_0 + f(t)$, with $f(t)$ defined as straight line segments

or a trigonometric function;

$\rho' = \rho'(t)$, values of $\rho'(t)$ being listed in tabular form.

The outputs from the code are n , inverse period, C_i , and F_m as functions of time.

A.2 RE 29, RE 129 codes

Both of these codes [54, 55] solve the system Eqs. (A-2) for $n(t)$ given k_{ex} . RE 29 is for the IBM-650 and RE 129 is the IBM-704 version.

Using the precursor equation C_i is eliminated from the neutron equation of Eqs. (A-2). The resulting equations are then:

$$\frac{dn}{dt} = \frac{k_{ex}n}{\Lambda} - \sum_{i=1}^I \frac{dC_i}{dt} + Q,$$

$$\frac{dC_i}{dt} = (1 + k_{ex})\beta_i \frac{n}{\Lambda} - \lambda_i C_i. \quad (A-4)$$

Both sides of Eqs. (A-4) are then integrated with respect to t , and the quantities under the integral signs are replaced by linear approximations valid in the time interval Δt under consideration. From the resulting expression, $n(t_j)$ is determined successively for a set of time values t_j .

Forms of k_{ex} available with the code are:

- (a) $k_{ex} = At + B \int_0^t n(y) dy$;
- (b) $k_{ex} = At + B [n(t) - 1]$ for $t < t_0$,
 $= B[n(t) - 1]$ for $t \geq t_0$;
- (c) k_{ex} specified at discrete time points;

$$(d) \quad k_{ex} = \sum_m \left\{ \Lambda_m (t - t_0)^m + \left[\int_0^t n(y) dy \right]^m \right\} \text{ and}$$

$$\Lambda = \sum_m L_m \left[\int_0^t n(y) dy \right]^m \text{ for } m = 0, 1, \dots, 4.$$

Forms (a), (b), and (c) are available with RE 29; forms (a), (c), and (d) are available with RE 129. For problems involving an initial step or ramp change in k_{ex} , an optional, artificial start routine is provided to compute the initial values of dC_i/dt and Δn (the variation of n over a time interval Δt).

The output of the code is n , $[n/(dn/dt)]$, k_{ex} and $\int_0^t n(y) dy$ for specified values of time t . In addition, for form (d) the computed neutron lifetime is printed.

Attempts to improve the above codes resulted in developing a new technique, "the method of collocation," and applying it [56] to the solution of the kinetics equations.

In this method Eqs. (A-4) are converted into an integral equation and a solution found by the method of collocation [57]. Included in the numerical scheme as applied to the kinetics equations is an automatic interval-size control.

The above method is the basis of an experimental code called COLLOREK [49].

A.3 RE 126, RE 135 codes

These codes [58] are in FORTRAN for the IBM-704. They solve Eqs. (A-2) for a specified reactivity step. The solution is accomplished by obtaining the roots s_j of the inhour equation and using these in the relation for the neutron density:

$$\frac{n(t)}{n_0} = \sum_{j=1}^{I+1} (A_j + B_j) \exp s_j t + C \quad (A-5)$$

A_j , B_j , and C are determined from λ_1 , β_1 , Λ , s_j , and $k_{ex,0}$, the form depending on whether the equilibrium or nonequilibrium precursor problem is treated.

RE 126 performs this calculation for the case of a reactivity step with equilibrium precursors. RE 135 computes the case of nonequilibrium precursors. In the latter computation the reactor is assumed to have been operating at a steady power n_0 for a buildup time of T_B seconds and this determines C_{i0} .

Both codes use an iterative procedure to determine the roots s_j . The problem is terminated when the difference between the reactivity as determined by the s_j and a specified reactivity is a minimum.

The output is: s_j , A_j , B_j , and also $\frac{1}{n_0} \frac{dn}{dt}$, $\frac{n}{n_0}$, $\frac{n}{dn/dt}$, and deviation from asymptotic period for each time value.

A.4 RE 138 code

This IBM-704 FORTRAN code [59] calculates k_{ex} and time derivatives of k_{ex} , using Eqs. (A-2), given the neutron density as a function of time. Defining:

$$\alpha(t) = \frac{1}{n} \frac{dn}{dt}$$

$$d_i(t) = \frac{\lambda_i \Lambda}{\beta_i} \frac{C_i}{n}, \quad (A-6)$$

evaluating the neutron density equation at t_j and t_{j+1} , and subtracting the result yields:

$$\Lambda \Delta \alpha_j = (1 - \beta) \Delta k_j + \sum_{i=1}^I \beta_i \Delta d_{ij}. \quad (A-7)$$

Assuming that the neutron density can be represented by

$$n = n_j \exp \bar{\alpha}_j (t - t_j) \text{ for } t_j \leq t \leq t_{j+1}, \quad (A-8)$$

where

$$\bar{\alpha}_j = \frac{1}{\Delta t_j} \ln \left(\frac{n_{j+1}}{n_j} \right)$$

and setting

$$a_j = \frac{\bar{\alpha}_{j-1} + \bar{\alpha}_j}{2}$$

the precursor equations can be integrated for Δd_{ij} . This is then substituted in the neutron formula to obtain Δk_j .

For each time value the code lists k_{ex} , dk_{ex}/dt , $d^2 k_{ex}/dt^2$, reactor period and $\int_0^t n(y) dy$.

A.5 RTS code

The RTS code [60] is for the IBM-704. In Eqs. (A-2) the C_i are eliminated and the Laplace Transform applied to the result. By means of convolution theorems and roots of auxiliary functions in the transform space the neutron density may be written as:

$$n(t) = n(0) + \sum_{j=0}^I A_j \times \int_0^t \exp [s_j(t-t')] k_{ex}(t') n(t') dt' + \Omega_0(t), \quad (A-9)$$

where A_j and s_j are universal constants for the fissionable species to be used for various values of Λ , and $\Omega_0(t)$ is a known function of time.

The RTS code solves Eq. (A-9) recursively at each time point for k_{ex} in the form

$$k_{ex} = P_1(t) + P_2(t) \int_0^t \exp R(t-t') n(t') dt',$$

where $P_1(t)$ is either a polynomial in t of specified degree, or may be given as a table; $P_2(t)$ is a polynomial in t of specified degree.

The code has an automatic interval size change procedure.

The quantities $n(t)$, $k_{ex}(t)$, $\int n dt$, $\Omega_0(t)$, $C_1(t)$ and other quantities are printed as output for each value of t .

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