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# INPUT-OUTPUT APPROACH TO OPTIMAL CONTROL

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The purpose of this paper is to present and illustrate two new methods for solving optimal control problems of systems represented by input-output relations. Many physical systems, such as nuclear reactor systems, are often studied by means of input-output measurements which, in general, result in input-output relations. It appears, therefore, that the study of optimal control problems through input-output methods has merit.

A simple example is used here to present the essentials of the methods even though they are applicable to complex linear[1] and to a class of nonlinear systems. Let the input-output relation be:

$$y(t) = \int_0^t h(t-s)\,u(s)\,ds \tag{1}$$

where u(t) is the input and y(t) is the output.

(I) Characteristic Expansion Method.[1] Let it be desired that the output be brought to $y_T$ at t=T and so that

$$J = \int_0^T u^2(t)\,dt \tag{2}$$

is minimized. Express the kernel h(T-t) as a finite series in terms of some characteristic orthonormal set, i. e.:

$$h(T-t) = \sum_{p=1}^{P} A_p \phi_p(t) \tag{3}$$

The orthogonality range is $[0, T]$ . Expand the input in terms of the same orthonormal set

$$u(t) = \sum_{p=1}^{\infty} B_p \phi_p(t) \tag{4}$$

Using (3) and (4) in (1) at t=T and in (2), find:

$$\text{(a)} \quad y_T = \sum_{p=1}^{P} A_p B_p \quad ; \quad \text{(b)} \quad J = \sum_{p=1}^{\infty} B_p^2 \tag{5}$$

Minimization of (5b), under constraint (5a), yields the complete set of $B_p$'s which define the optimal input.

(II) Functional Analysis Method. Suppose it is desired to transfer the output to $y_T$ in a minimum time, with the input constrained by

$$\|u(t)\|_p = \left(\int_0^T |u(t)|^p dt\right)^{\frac{1}{p}} \leqslant K \tag{6}$$

Apply Hölder's inequality to (1) at t=T to find:

$$\text{(a)} \quad y_T \leqslant \|u(t)\|_p \|h(T-t)\|_q \text{ or (b)} \quad K \geqslant \frac{|y_T|}{\|h(T-t)\|_q} \tag{7}$$

where $1/p + 1/q = 1$. Since $\|h(T-t)\|_q$ is monotonically increasing with T, its minimum implies minimum T. In order for $\|h(T-t)\|_q$ to be minimum, (7) must be satisfied with the equality signs. Thus, the optimal input is

$$u(t) = K^q \left|\frac{h(T^*-t)}{y_T}\right|^{q-1} \operatorname{sgn}\left\{\frac{h(T^*-t)}{y_T}\right\} \tag{8}$$

where the minimum time T* is found from (7). The preceding results are identical to those of other methods based on differential equations.[2]

The present methods have been applied to several systems. The following example illustrates the results. A simple expression for the average incremental temperature, $\theta(t)$, of a reactor is given by[3]:

$$\theta(t) = a \int_0^\infty \sin b(t-s)\, u(s)\, ds + c \int_0^\infty \int_0^\infty \sin b(t-s_1) \cos b(s_1 - s_2)\, u(s_1)\, u(s_2)\, ds_1\, ds_2 \quad (9)$$

if terms only up to second order are retained. The constants a, b, and c depend on the reactor parameters, and $u(t)$ is the externally controlled reactivity. Let it be desired that $\theta(t)$ be brought to $\theta_T$ at $t=T$ with the minimum value of the integral of $u^2(t)$. Minimization of the integral is equivalent to penalizing small periods. Suppose the reactor is at equilibrium for $t < 0$. Set $T=2\pi/b$ for simplicity and find that the common characteristic orthonormal functions to all kernels are: $\phi_1(t)=\sqrt{1/T}$ ; $\phi_{2n}(t)=\sqrt{2/T}\,\sin(2\pi nt/T)$ ; $\phi_{2n+1}(t)=\sqrt{2/T}\,\cos(2\pi nt/T)$ ; $n=1, 2, \ldots$ Use Method I to conclude that, when $\theta_T \ll a/\sqrt{2\pi b}$, the optimal input is:

$$u(t) = \sum_{p=1}^{5} B_p \phi_p(t) \quad (10)$$

where $B_1 = \frac{(1-\sqrt{2})a\sqrt{b}}{5\sqrt{2\pi}\,c}$, $B_2 = \frac{5\sqrt{b}}{(\sqrt{2}-5)\sqrt{\pi}\,a}$, $B_3=B_4=0$, $B_5 = \frac{a\sqrt{b}}{5\sqrt{\pi}\,c}$.

Next consider the same system without the nonlinear term. Suppose it is desired to bring $\theta(t)$ to $\theta_T$ in minimum time with the input constrained by

$$\max_t \left| u(t) \right| \leq 1 \quad (11)$$

Using (8) for q=1 and K=1, find that the optimal input is:

$$u(t) = \text{sgn}\left\{\frac{a \sin b(T^* - t)}{\theta_T}\right\} \tag{12}$$

where T* is obtained by satisfying

$$\frac{|\theta_T|}{\| a \sin b(T^* - t)\|_1} = 1 \tag{13}$$

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