47

Some Applications of Mathematical Methods to Nuclear Engineering at the Massachusetts Institute of Technology*

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INTRODUCTION

The purpose of this paper is to discuss some applications of mathematical methods to nuclear engineering at the Massachusetts Institute of Technology (MIT). The word *some* in the title is used advisedly because in the limited space available I will not be able to do justice to all activities of this nature in our department. I will restrict my remarks to problems that concern us and fall within the main themes of this conference.

Specifically, I would like to discuss briefly (a) a new criterion of asymptotic stability for nuclear reactors described by nonlinear equations including all delayed-neutron precursors; (b) a simple example illustrating the procedure for deriving stability criteria by investigating the nature of equilibrium states at infinity; (c) a novel approach to problems of nonlinear optimum control for systems described by means of input–output functional relations; (d) some results on questions of space–time nuclear reactor dynamics; (e) an application of dynamic programming to the problem of optimum refueling of a power nuclear reactor.

*This is an invited review paper.

A NEW CRITERION OF ASYMPTOTIC STABILITY

General Remarks

It has been shown that stability analyses of nonlinear point reactor kinetics, resulting in criteria that do not include the delayed-neutron precursor constants, are either overrestrictive or nonconservative. For example, in cases where the feedback transfer function is a lagging function of real frequencies, the analysis is overrestrictive because delayed neutrons may relax substantially the requirements for stability.[1] On the other hand, when the feedback transfer function is a leading function of real frequencies, the analysis is nonconservative because the reactor model without delayed neutrons may be absolutely stable at all operating power levels, whereas the model with delayed neutrons may be linearly stable only for a limited power range.[2]

In addition, it has been recognized that criteria which guarantee absolute asymptotic stability are impractical.[1] The impracticality arises from the fact that no real reactor can be operated at very high power levels.

These observations suggest that physically meaningful stability requirements should always include the delayed-neutron precursor parameters and a finite range of operating power levels. In what follows, a new criterion of stability is presented which incorporates the preceding suggestions. To the best of my knowledge, this criterion is the most general and least restrictive stability requirement that has been derived to date.

Reactor Model

If it is assumed that the reactor admits a unique equilibrium power level, P_1, for a given reactivity input, the kinetics equations for $t > 0$ can be written in terms of normalized, dimensionless, and incremental variables as

$$\frac{dp(t)}{dt} = -\sum_{i}^{m} \frac{\beta_i}{\Lambda} [p(t) - c_i(t)] + k(t) \tag{1}$$

$$\frac{dc_i(t)}{dt} = \lambda_i [p(t) - c_i(t)] \qquad (i = 1, 2, \ldots, m) \tag{2}$$

$$k(t) = -\frac{P_1}{\Lambda} [1 + p(t)] \left\{ \int_{-\infty}^{t} f(t - \tau)\, p(\tau)\, d\tau + F_2 [p(\tau)] \right\} \tag{3}$$

$$f(t) = 0 \qquad (t < 0) \tag{4}$$

The minimum physical value of the variables is $p = c_i = -1$. The meaning of Eq. 3 is that, to a first approximation, feedback effects are expressed as a linear convolution of the normalized incremental power $[\int_{-\infty}^{t} f(t-\tau)\, p(\tau)\, d\tau]$. Any higher order effects arising either from nonlinear feedback per se or from variations of the values of β_i, λ_i, and Λ during a transient are included in the nonlinear functional $F_2[p(\tau)]$. The properties and implications of this functional have been discussed elsewhere.[1]

Before the stability analysis is discussed, it is convenient to define the following:

$$q(j\omega,t) = \int_{-\infty}^{t} e^{j\omega(t-\tau)}\, k(\tau)\, d\tau \tag{5}$$

$$q^*(j\omega,t) = \int_{-\infty}^{t} e^{-j\omega(t-\tau)}\, k(\tau)\, d\tau \tag{6}$$

$$R(s) = \left(s + \sum_{i}^{m} \frac{\beta_i}{\Lambda} \frac{s}{s+\lambda_i}\right)^{-1} = \text{zero-power-reactor transfer function} \tag{7}$$

$$r(t) = \text{inverse transform of } R(s) \qquad [r(t) = 0] \quad (t < 0) \tag{8}$$

$$F(s) = \text{transform of } f(t) = \text{feedback transfer function} \tag{9}$$

With these definitions Eqs. 1 to 3 yield

$$p(t) = \int_{-\infty}^{t} r(t-\tau)\, k(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(j\omega)\, q(j\omega,t)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} R^*(j\omega)\, q^*(j\omega,t)\, d\omega \tag{10}$$

$$\int_{-\infty}^{t} f(t-\tau)\, p(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, R(j\omega)\, q(j\omega,t)\, d\omega \tag{11}$$

$$k(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q(j\omega,t)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} q^*(j\omega,t)\, d\omega \tag{12}$$

The derivation of Eqs. 10 to 12 is straightforward. It requires certain changes of order of integration which imply that

$$\int_{-\infty}^{0} |p(\tau)|\, d\tau < \infty \qquad \int_{-\infty}^{0} |k(\tau)|\, d\tau < \infty \tag{13}$$

Conditions for Stability of the Linear Approximation

For the linear approximation of Eqs. 1 to 3 to be stable, the roots of the characteristic equation

$$1 + \frac{P_1}{\Lambda} R(s)\, F(s) = 0 \tag{14}$$

must lie in the left-half complex plane. It is assumed that this is true.

It is further assumed that beyond a certain critical power level, P_c, one or more of the roots of Eq. 14 move into the right-half complex plane and that the reactor becomes linearly unstable. In other words, for

$$P_c = a_c P_1 \qquad (a_c > 1) \tag{15}$$

the equation

$$1 + \frac{a_c P_1}{\Lambda} R(s)\, F(s) = 0 \tag{16}$$

admits roots on the $j\omega$-axis. The number a_c is a measure of the margin of linear stability with respect to the operating power level, P_1.

Conditions for Nonlinear Asymptotic Stability

Let $F_2[p(\tau)] \equiv 0$. In other words, assume that

$$k(t) = -\frac{P_1}{\Lambda}[1 + p(t)] \int_{-\infty}^{t} f(t - \tau)\, p(\tau)\, d\tau \tag{17}$$

Note that this assumption does not alter any of the preceding results.

Consider the scalar function

$$V = p(t) - \ln[1 + p(t)] - \frac{1}{2d^2} p^2(t)$$

$$+ \sum_i^m \frac{\beta_i}{\lambda_i \Lambda}\left[c_i(t) - \ln[1 + c_i(t)] - \frac{1}{2d^2} c_i^2(t)\right]$$

$$+ \sum_i^m \frac{\beta_i}{\Lambda} \int_{-\infty}^{t} [p(\tau) - c_i(\tau)]^2 \left[\frac{1}{[1 + p(\tau)][1 + c_i(\tau)]} - \frac{1}{d^2}\right] d\tau$$

$$+\frac{b\Lambda}{a}\int_{-\infty}^{t}\frac{a-1-p(\tau)}{1+p(\tau)}\,k^2(\tau)\,d\tau \tag{18}$$

where a, b, and d are positive numbers to be determined (a, d > 1). If $d \geq a$, the function V is positive definite in the region

$$d \geq a \qquad -1 < p < a-1 \qquad -1 < c_i < a-1 \qquad (i = 1, 2, \ldots, m) \tag{19a}$$

If d < a, the function V is positive definite in the region

$$d < a \qquad -1 < p < d-1 \qquad -1 < c_i < d-1 \qquad (i = 1, 2, \ldots, m) \tag{19b}$$

In addition, the function V admits continuous partial derivatives with respect to p(t) and all $c_i(t)$; and V is equal to zero only for $p(t) = c_i(t) = 0$.

The time derivative of V along the trajectories of the system of Eqs. 1, 2, and 17 is

$$\frac{dV}{dt} = -\frac{P_1}{\Lambda}\,p(t)\int_{-\infty}^{t} f(t-\tau)\,p(\tau)\,d\tau - \frac{1}{d^2}\,p(t)\,k(t)$$
$$-\frac{b\Lambda}{a}\,k^2(t) - bP_1\,k(t)\int_{-\infty}^{t} f(t-\tau)\,p(\tau)\,d\tau \tag{20}$$

If all the terms in the right-hand side of Eq. 20 are written as a function of $q(j\omega,t)$ by means of Eqs. 10 to 12, then it is found that

$$\frac{dV}{dt} = \mp\frac{1}{4\pi^2}\left|\int_{-\infty}^{\infty}[\pm\mathrm{Re}\;G(j\omega)]^{1/2} q(j\omega,t)\,d\omega\right|^2 \tag{21}$$

where

$$G(j\omega) = \frac{P_1}{\Lambda}\,|R(j\omega)|^2\,F(j\omega) + \frac{R^*(j\omega)}{d^2} + \frac{b\Lambda}{a}\left[1 + \frac{aP_1}{\Lambda}\,R(j\omega)\,F(j\omega)\right] \tag{22}$$

and

$$d^2 = a$$

A sufficient condition for this derivative to be negative definite is

$$\mathrm{Re}\;G(j\omega) > 0 \tag{23}$$

Consequently, if there exist numbers a and b such that condition 23 is satisfied, the solutions of Eqs. 1, 2, and 17 are asymptotically stable because V is a Liapunov function with a negative definite time derivative. In other words, given a reactor described by Eqs. 1, 2, and 17, the operating power level, P_1, is asymptotically stable with respect to all initial perturbations that lie in the region (total quantities)

$$d^2 = a \qquad P < aP_1 \qquad C_i < aC_{1i} \qquad (i = 1, 2, \ldots, m) \tag{24}$$

provided that there exist positive numbers a and b which make the function G(s) a positive real function without zeros on the $j\omega$-axis.

It is evident from the discussion on linear stability that the number a must be

$$a < a_c \tag{25}$$

Its exact value as well as the exact value of b depends on the particular form of the feedback transfer function.

The sufficient condition 23 can be written in a simpler form when $d^2 = a$. Indeed, then

$$G(j\omega) = \left[\frac{R^*(j\omega)}{a} + \frac{b\Lambda}{a}\right]\left[1 + \frac{aP_1}{\Lambda} R(j\omega)\, F(j\omega)\right]$$

$$= \frac{1}{a}\left|1 + \frac{aP_1}{\Lambda} R(j\omega)\, F(j\omega)\right|^2 \frac{R^*(j\omega) + b\Lambda}{1 + (aP_1/\Lambda)\, R^*(j\omega)\, F^*(j\omega)} \tag{26}$$

Therefore condition 23 is equivalent to

$$\mathrm{Re}\left[\frac{R(j\omega) + b\Lambda}{1 + (aP_1/\Lambda)\, R(j\omega)\, F(j\omega)}\right] > 0 \tag{27}$$

Another simplified sufficient criterion is derived by taking $d^2 = a$ and $b = 0$. Then condition 23 becomes

$$\mathrm{Re}\left[\frac{R(j\omega)}{1 + (aP_1/\Lambda)\, R(j\omega)\, F(j\omega)}\right] > 0 \tag{28}$$

In other words, the reactor is asymptotically stable with respect to all initial perturbations in the region 24 if the reactor transfer function at power is a positive real function without zeros on the $j\omega$-axis.

A similar procedure can be used for the case where $F_2[p(\tau)] \neq 0$, i.e., when k(t) is given by Eq. 3. For example, consider the function V in Eq. 18. Its time derivative along the trajectories of Eqs. 1 to 3 is

$$\frac{dV}{dt} = -\frac{P_1}{\Lambda} p(t) \int_{-\infty}^{t} f(t-\tau)\, p(\tau)\, d\tau - \frac{1}{d^2} p(t)\, k(t) - \frac{b\Lambda}{a} k^2(t)$$

$$- bP_1\, k(t) \int_{-\infty}^{t} f(t-\tau)\, p(\tau)\, d\tau + F_2[p(\tau)]\,[p(t) + b\Lambda\, k(t)] \qquad (29)$$

If it is assumed that $F_2[p(\tau)]$ has always the opposite sign from $p(t) + b\Lambda\, k(t)$ and that condition 23 is satisfied, then the time derivative given by Eq. 29 is negative definite, and the reactor is asymptotically stable in the region defined by inequalities 24. Other sufficient conditions can be derived for different specifications on $F_2[p(\tau)]$.

Also, a sufficient condition for Lagrangian stability of the solutions of Eqs. 1 to 3 is given in Ref. 1. Specifically, the solutions of Eqs. 1 to 3 are Lagrange stable with respect to all initial perturbations if there exists a positive number a such that

$$\mathrm{Re}\left[\frac{R(j\omega)}{1 + (aP_1/\Lambda)\, R(j\omega)\, F(j\omega)}\right] > 0 \qquad [f(t) < 0] \qquad (30)$$

Comparisons with Existing Criteria for the Case $F_2[p(\tau)] \equiv 0$

Welton's sufficient criterion[16] $\mathrm{Re}\, F(j\omega) \gtrsim 0$ is a special case of condition 23 for $a = d = \infty$ and $b = 0$. It implies that the reactor is asymptotically stable for an infinite range of operating power levels ($a_c = \infty$), and it does not include the delayed-neutron precursor constants. It is evident that, even if the feedback transfer function, F(s), does not satisfy Welton's criterion, it can satisfy inequality 23.

Popov[3] derived a sufficient criterion for asymptotic stability by means of certain inequalities. In terms of the present nomenclature, this criterion is

$$\mathrm{Re}\left\{\frac{P_1}{\Lambda} |R(j\omega)|^2\, F(j\omega) + \frac{b\Lambda}{a}\left[1 + \frac{aP_1}{\Lambda} R(j\omega)\, F(j\omega)\right]\right\} > 0 \qquad (31)$$

The range of acceptable initial perturbations is given in an implicit form. Popov's criterion is a special case of condition 23 for $d = \infty$. It is evident that the criterion derived in this paper is less restrictive than

Popov's criterion since the reactor transfer function at zero power, R(s), is a positive real function [Re R(jω) = Re R*(jω) ≥ 0].

It is concluded that the sufficient condition for asymptotic stability stated by inequality 23 is the most general criterion that has been derived to date.

It should be noted that all the sufficient criteria for nonlinear stability discussed so far involve only parameters characteristic of the linear approximation of Eqs. 1 to 3. This is an advantage both because these parameters can be identified experimentally and because procedures for characterizing nonlinear systems are not yet fully developed.

STABILITY CRITERIA DERIVED FROM INVESTIGATIONS OF SINGULARITIES AT INFINITY

General Remarks

Next, I would like to discuss a novel technique for deriving necessary and sufficient criteria for stability of solutions of ordinary differential equations with polynomial nonlinearities. This technique requires that all singularities at infinity be locally unstable. This comes about because all trajectories of a dynamical system begin from and end at singularities. One of my students, Miguel Barandiaran, worked on this problem for his doctoral thesis.[4]

A three-dimensional example is treated in Ref. 5. To bring out the salient features of the technique without excessive mathematical complexities, I will describe another problem with two state variables.

Consider the set of differential equations:

$$\frac{d\Phi}{dt} = \frac{1}{\tau_e}\left[\delta_o - \frac{\sigma_x X}{c\sigma_f}\right]\Phi \tag{32}$$

$$\frac{dX}{dt} = y_x \sigma_f \Phi - \lambda_x X - \sigma_x \Phi X \tag{33}$$

where c, λ_x, σ_f, σ_x, τ_e, and y_x are positive constants and δ_o is a positive or a negative constant. For a physical interpretation of Eqs. 32 and 33, see Ref. 6. The region of physical interest is Φ, $X > 0$. The purpose of this analysis is to derive necessary and sufficient conditions that must be satisfied by the constants of the system so that the solutions are asymptotically stable or bounded with respect to all physically realizable initial conditions. For completeness, singularities both in the finite phase plane and at infinity are examined.

Equilibrium States in the Finite Phase Plane

There are two equilibrium states in the finite plane:

$$\Phi_o = 0 \qquad X_o = 0$$

and

$$\Phi_1 = \frac{c\delta_o\lambda_x}{\sigma_x(y_x - c\delta_o)} \qquad X_1 = \frac{c\delta_o\sigma_f}{\sigma_x}$$

The equilibrium state (Φ_1, X_1) is physically meaningful only when

$$0 < c\delta_o < y_x \tag{34}$$

The eigenvalues associated with the equilibrium state at the origin are

$$s_1 = -\lambda_x \qquad s_2 = \frac{\delta_o}{\tau_e} \tag{35}$$

The first eigenvalue corresponds to the vertical direction $\Phi = 0$. The second eigenvalue corresponds to an eigenvector of slope $y_x\sigma_f\tau_e/(\delta_o + \lambda_x\tau_e)$. For $\delta_o > 0$, the origin is a saddle point, and, for $\delta_o < 0$, it is a stable node. The origin becomes structurally unstable[7] for $\delta_o = 0$ and $\delta_o = -\lambda_x\tau_e$.

The eigenvalues at the second equilibrium state are given by the characteristic equation:

$$s^2 + \frac{\lambda_x y_x}{y_x - c\delta_o}\, s + \frac{\delta_o\lambda_x}{\tau_e} = 0 \tag{36}$$

For $\delta_o < 0$, these eigenvalues are of opposite sign, and the equilibrium point is of the saddle type and nonphysical. For $\delta_o > 0$, the eigenvalues are real or complex, and the equilibrium state becomes a focus or a node depending on whether

$$\frac{4\delta_o}{\lambda_x\tau_e}\left[1 - \frac{c\delta_o}{y_x}\right]^2 \gtrless 1 \tag{37}$$

respectively. The equilibrium point is stable for $c\delta_o < y_x$.

Singularities at Infinity

To establish the singularities at infinity, the state variables are expressed in terms of homogeneous coordinates, and the time variable is eliminated from Eqs. 32 and 33. More specifically, if

$$\Phi = \frac{u}{z} \qquad X = \frac{v}{z} \tag{38}$$

the points at infinity correspond to $z = 0$, and, without the time variable, Eqs. 32 and 33 can be written in the form

$$\begin{vmatrix} du & dv & dz \\ u & v & z \\ \frac{\delta_o}{\tau_e} uz - \frac{\sigma_x}{c\sigma_f\tau_e} uv & y_x\sigma_x uz - \lambda_x vz - \sigma_x uv & 0 \end{vmatrix} = 0 \tag{39}$$

For this equation to be satisfied for $z = 0$, regardless of the direction (du, dv, dz), either the elements of the second and third rows are proportional or the last row is zero. Therefore the following singularities exist at infinity:

1. Two points along the direction

$$\frac{v}{u} = \frac{X}{\Phi} = c\sigma_f\tau_e \qquad (z = 0)$$

2. Two double points along the vertical direction

$$u = \Phi = 0 \qquad (z = 0)$$

3. Two double points along the horizontal direction

$$v = X = 0 \qquad (z = 0)$$

For examination of the eigenvalues at the points along the direction $v/u = c\sigma_f\tau_e$, it is expedient to transfer these points to some convenient point in finite space without altering the eigenvalues. It can be shown that this can be readily done by means of a singular projective transformation.[4] More specifically, if the singular transformation

$$w' = \frac{u}{u} = 1 \qquad v' = \frac{v}{u} \qquad z' = \frac{z}{u} \tag{40}$$

is chosen, then Eq. 39 is represented in the two-dimensional Cartesian space (v',z') by the equation

$$\frac{dv'}{y_x\sigma_f z' - (\lambda_x + \frac{\delta_o}{\tau_e})v'z' - \sigma_x v' + \frac{\sigma_x}{c\sigma_f\tau_e}v'^2} = \frac{dz'}{-\frac{\delta_o}{\tau_e}z'^2 + \frac{\sigma_x}{c\sigma_f\tau_e}v'z'} \qquad (41)$$

and the singularity is at the point

$$v' = c\sigma_f\tau_e \qquad z' = 0 \qquad (42)$$

If the origin of the (v',z') plane is transferred to the singularity (Eq. 42), then Eq. 41 becomes

$$\frac{dv''}{\sigma_x v'' + y_x\sigma_f - (\lambda_x + \frac{\delta_o}{\tau_e})\, c\sigma_f\tau_e z' + O_2(v'',z')} = \frac{dz'}{\sigma_x z' + O_2(v'',z')} \qquad (43)$$

where $v'' = v' - c\sigma_f\tau_e$ and $O_2(v'',z')$ are second-order polynomials in (v'',z'). It is evident that the eigenvalues are

$$s_1 = s_2 = \sigma_x \qquad (44)$$

Hence the singularity is always of the unstable star type.

The character of the double singularities along the vertical direction can be deduced by an examination of the behavior of the trajectories in the vicinity of these points. Note first that the two straight line trajectories $\Phi = 0$ and $z = 0$ go through these singularities and act as separatrices. In the vicinity of the separatrices, the trajectories behave as shown in Fig. 1, as can be readily deduced from Eqs. 32 and 33 evaluated for very small Φ and very large X. Consequently one of the double singularities is an unstable saddle, whereas the other is an unstable node for all values of the system parameters.

Finally, the double singularities along the horizontal direction can be investigated in a similar fashion. Consider first the singularities for $\Phi > 0$. The straight line trajectory $z = 0$ is a separatrix for this point also. The other separatrix is a curve asymptotically tangent to the line

$$X = \frac{y_x\sigma_f}{\sigma_x} \qquad (45)$$

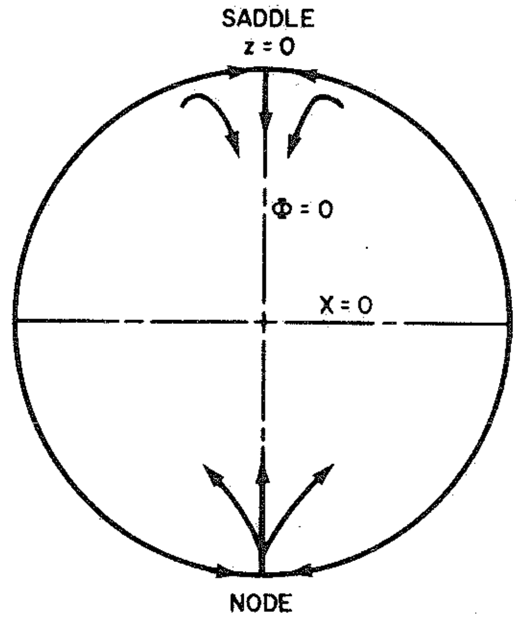


Fig. 1 – Behavior of trajectories in the vicinity of the double singularities at infinity along the direction $\Phi = 0$.

This line is also the asymptote to the 0° isocline. The nature of the singularity depends on the relative position of the 0° and the 90° isoclines. The 90° isocline is

$$X = \frac{c\delta_o \sigma_f}{\sigma_x} \tag{46}$$

The two possible positions of the isoclines and the corresponding behavior of the trajectories are shown in Fig. 2. If $c\delta_o < y_x$, the singularity is of the saddle type. If $c\delta_o > y_x$, the singularity is of the stable node type. For $c\delta_o = y_x$, the singular point (Φ_1, X_1) goes to infinity, and the singularity along the direction $X = 0$ becomes triple but maintains its nodal character because the 0° isocline remains below the 90° isocline. Thus it is concluded that the singularity along the positive Φ direction is unstable for $c\delta_o < y_x$ and stable for $c\delta_o \geq y_x$. Similar observations can be made for the singularity at $\Phi < 0$.

When the results of the investigations of all the singularities are combined, it is concluded that the necessary and sufficient condition for boundedness of all physically meaningful solutions, i.e., the necessary and sufficient condition for the physically meaningful equilibrium states at infinity to be locally unstable, is that

$$c\delta_o < y_x \tag{47}$$

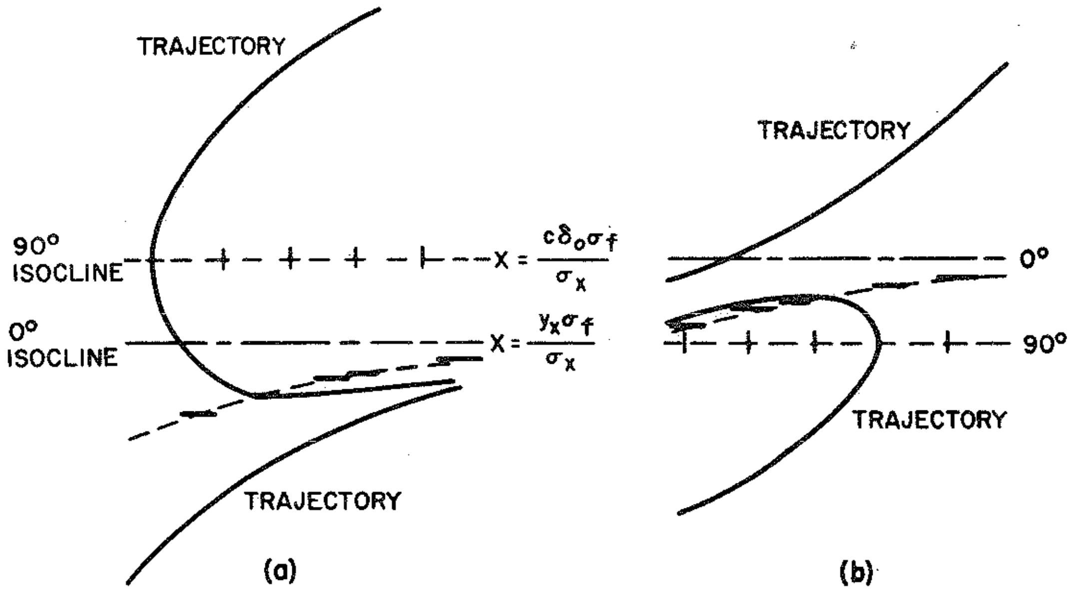


Fig. 2 – Behavior of trajectories in the vicinity of the double singularity along the direction $X = 0$ ($\Phi > 0$): (a) when the 90° isocline is above the 0° isocline, i.e., when $c\delta_o \gtrsim y_x$, and (b) when the 0° isocline is above the 90° isocline, i.e., $c\delta_o < y_x$.

In fact, when this condition is satisfied, the solutions are also asymptotically stable because there are no limit cycles and the equilibrium states in the finite plane are asymptotically stable. It is important to emphasize that these conclusions pertaining to a nonlinear system have been derived by considering only local (linear) properties of the singularities.

To show the topology of the trajectories, a representative phase portrait for each of the regions of the parameters in which the system is structurally stable (see Fig. 3) is given in Figs. 4 to 7. The meaning of these portraits is self-explanatory.

INPUT–OUTPUT APPROACH TO OPTIMUM CONTROL

General Remarks

Ordinarily, optimum control problems are stated in terms of differential equations. A typical statement may be made: Given a system described by the set of differential equations

$$\frac{d}{dt} y(t) = f[y(t),u(t)] \tag{48}$$

where $\mathbf{y}(t)$ = an n-state vector

$\mathbf{y}(t_0) = \eta_0$

$\mathbf{u}(t)$ = an r-control vector

find the optimum control $\mathbf{u}(t)$, $t \in [t_0, t_1]$, for which an extremum is attained by the cost functional

$$J[\mathbf{u}(t)] = \int_{t_0}^{t} L[\mathbf{y}(t), \mathbf{u}(t)]\, dt \tag{49}$$

when $\mathbf{u}(t)$ belongs to some constrained or unconstrained space

$\mathbf{y}(t_1) = \eta$

L = a specified operator

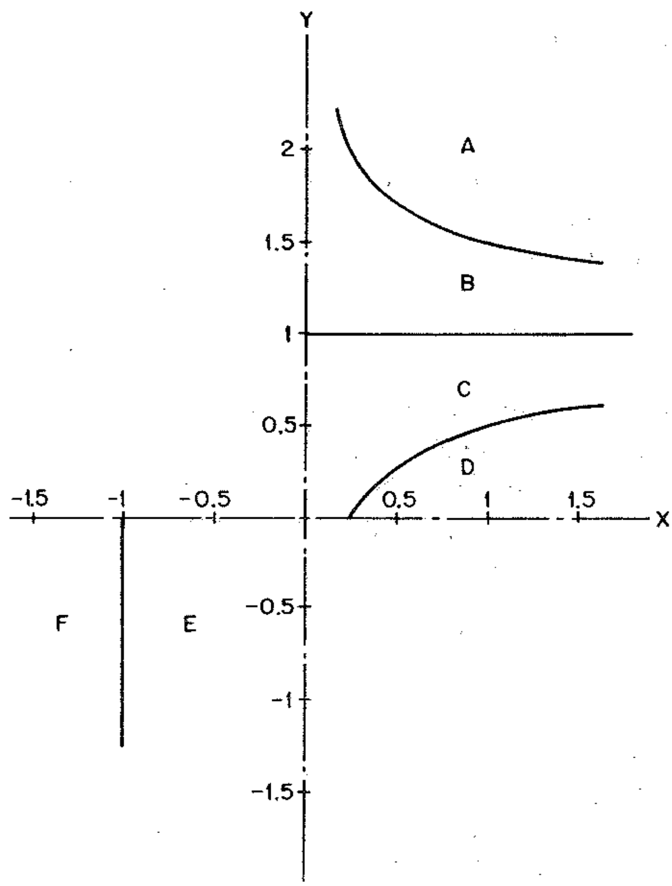


Fig. 3 – Regions of structural stability determined in terms of the parameters $x = \delta_o/\lambda_x\tau_e$ and $y = c\delta_o/y_x$ and by means of Eq. 37 and conditions $\delta_o = 0$ and $\delta_o = -\lambda_x\tau_e$.

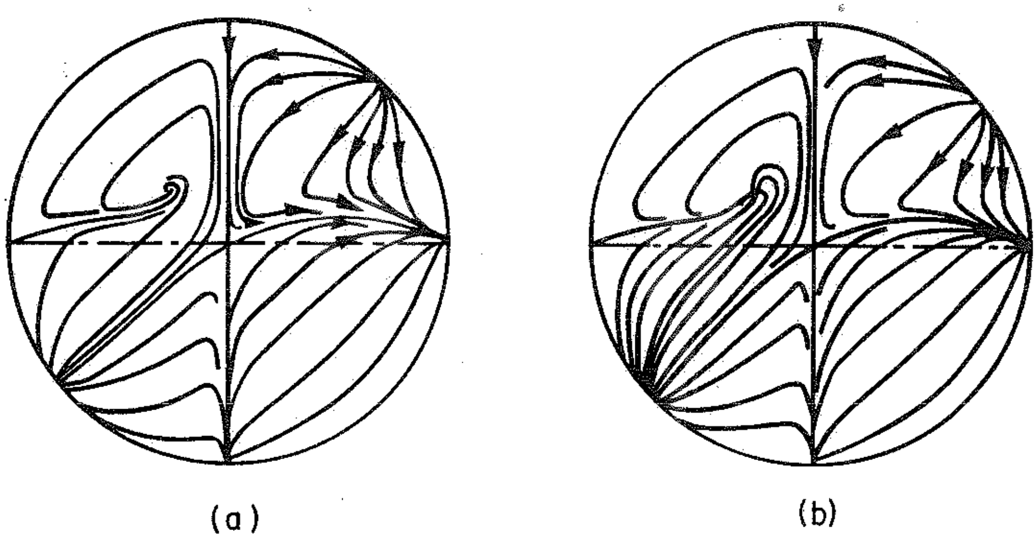


Fig. 4 – Typical behavior of trajectories for parameter values lying in (a) region A and (b) region B of Fig. 3.

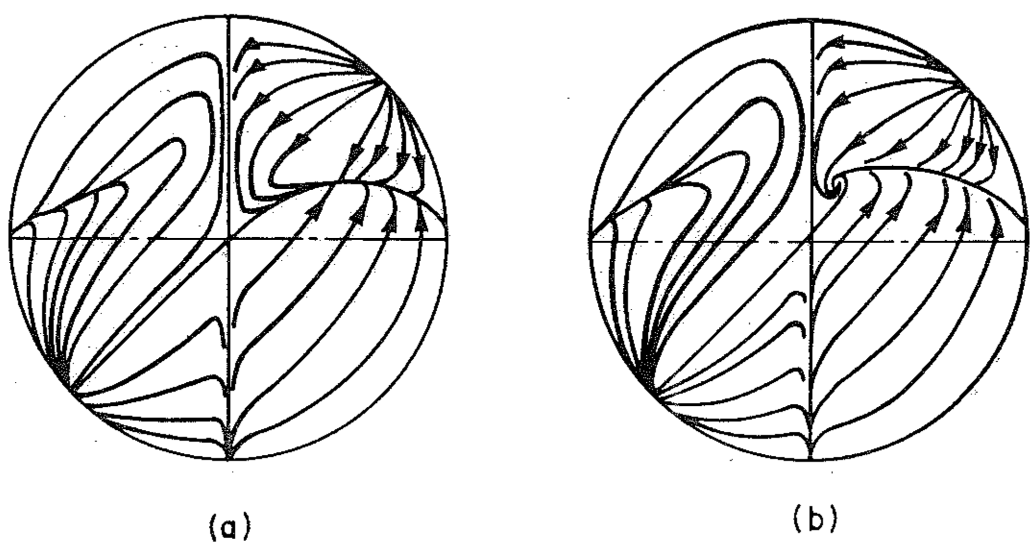
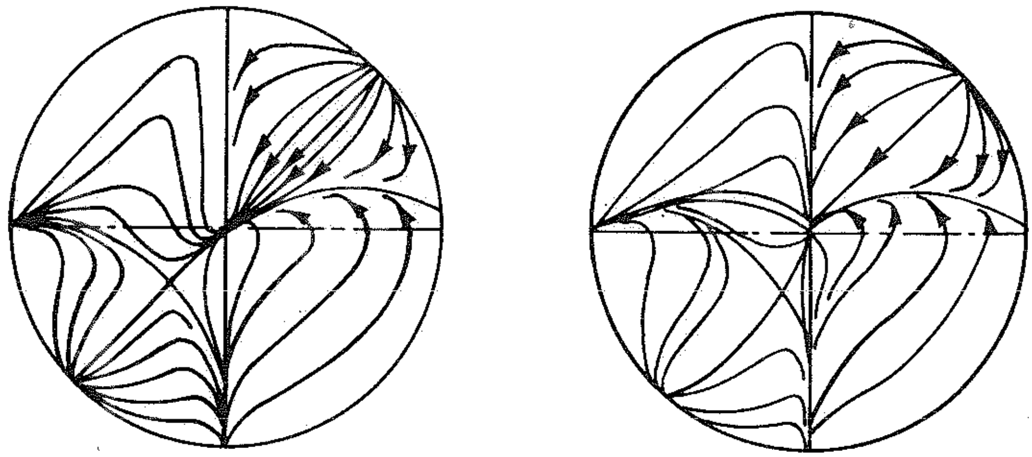
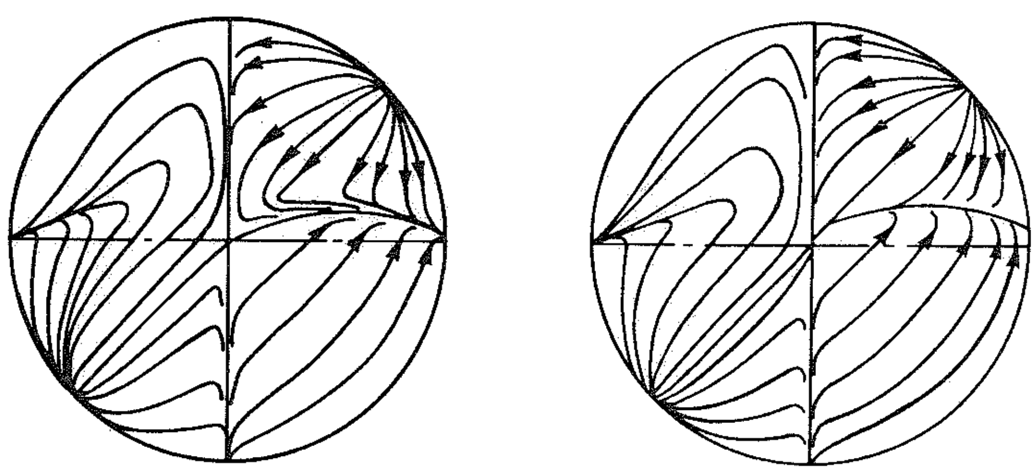


Fig. 5 – Typical behavior of trajectories for parameter values lying in (a) region C and (b) region D of Fig. 3.



(a) (b)

Fig. 6 – Typical behavior of trajectories for parameter values lying in (a) region E and (b) region F of Fig. 3.



(a) (b)

Fig. 7 – Typical behavior of trajectories for parameter values lying on (a) the boundary of regions B and C and (b) the boundary of regions C and E of Fig. 3.

Another way of looking at this problem is to consider the control vector as an input and the state vector as an output. Thus the system may be represented as an input—output relation:

$$\mathbf{y}(t) = \mathbf{H}[\mathbf{u}(t)] \tag{50}$$

where $\mathbf{H}$ is, in general, a nonlinear operator. Such representations have been discussed in Refs. 8 and 9. Under these conditions the cost functional is an explicit function of the input:

$$J[\mathbf{u}(t)] = \int_{t_0}^{t_1} L\{\mathbf{H}[\mathbf{u}(t),\mathbf{u}(t)]\}\, dt \tag{51}$$

The explicit dependence of the cost functional on $\mathbf{u}(t)$ may facilitate the analysis of the control problem at hand, particularly when $\mathbf{H}$ is nonlinear. It is for this reason that one of my students, Sang H. Kyong, is currently investigating optimum control of nonlinear systems described by input—output relations of the form given by Eq. 50. I will not discuss all the details of this work. To illustrate its salient features, however, I will present two simple examples.

A Linear Optimum Control Problem

Consider the dynamical system

$$\mathbf{y}(t) = \int_{t_0}^{t} [h_{ik}(t,\sigma)]\, \mathbf{u}(\sigma)\, d\sigma \tag{52}$$

where $[h_{ik}(t,\sigma)]$ is a matrix conformable with $\mathbf{u}(\sigma)$ and with elements $h_{ik}(t,\sigma)$. Suppose that it is desired to bring the output vector to $\boldsymbol{\eta}$ at time t_1, subject to the constraint that the cost

$$J_E = \int_{t_0}^{t_1} \sum_{i=1}^{r} u_i^2(t)\, dt \tag{53}$$

be minimum, where $u_i(t)$ is the *i*th component of $\mathbf{u}(t)$. No restrictions are imposed on the magnitude of $\mathbf{u}(t)$.

At the terminal time, Eq. 52 becomes

$$\boldsymbol{\eta} = \int_{t_0}^{t_1} [h_{ik}(t_1,\sigma)]\, \mathbf{u}(\sigma)\, d\sigma \tag{54}$$

To proceed, assume that all the elements of each row of the matrix $[h_{ik}(t_1,\sigma)]$ are of the same functional character. In other words, for a given i assume that each element $h_{ik}(t_1,\sigma)$ can be expanded into a finite series in terms of the members of a characteristic, complete, and orthonormal set, $\{\phi_n^k(\sigma)\}$:

$$h_{ik}(t_1,\sigma) = \sum_{\mu=1}^{m^{ik}} A_\mu^{ik}\,\phi_\mu^k(\sigma) \qquad (m^{ik} < \infty) \tag{55}$$

The orthonormality interval is always $[t_0,t_1]$. In addition, expand each component of $\mathbf{u}(\sigma)$ into an infinite series:

$$u_k(\sigma) = \sum_{\mu=1}^{\infty} B_\mu^k\,\phi_\mu^k(\sigma) \tag{56}$$

Substitution of Eqs. 55 and 56 into Eq. 54 yields

$$\begin{aligned} \eta_1 &= \sum_{k=1}^{r}\sum_{\mu=1}^{m^{ik}} A_\mu^{ik}B_\mu^k \\ \eta_2 &= \sum_{k=1}^{r}\sum_{\mu=1}^{m^{2k}} A_\mu^{2k}B_\mu^k \\ &\text{to} \\ \eta_n &= \sum_{k=1}^{r}\sum_{\mu=1}^{m^{nk}} A_\mu^{nk}B_\mu^k \end{aligned} \tag{57}$$

Inputs of the form of Eq. 56 satisfy the system and the terminal conditions exactly if the coefficients B_μ^k ($k = 1, 2, \ldots, r$ and $\mu = 1, 2, \ldots, m_k$) satisfy Eqs. 57 and if the coefficients B_μ^k ($\mu > m_k$ and $k = 1, 2, \ldots, r$) have arbitrary values, where m_k is the largest of the numbers m^{ik} for a given k and $i = 1, 2, \ldots, n$. The number of coefficients that can be determined from Eqs. 57 is

$$m = \sum_{k=1}^{r} m_k \tag{58}$$

Therefore three distinct cases can arise:

1. $m < n$. Since the n-relations of Eqs. 57 are linearly independent, it is impossible to satisfy them. In other words, there is no control that accomplishes the desired task.

2. $m = n$. In this case m coefficients can be uniquely determined from Eqs. 57.

3. $m > n$. This is the most usual case encountered in practice. Here n coefficients can be determined in terms of m–n others, i.e.,

$$B_r = f_r(B_{n+1}, \ldots, B_m) \qquad (r = 1, 2, \ldots, n) \tag{59}$$

where the coefficients B^k_μ have been reordered by means of a single index r.

Next, to complete the solution, consider the cost constraint. If Eq. 56 is replaced in Eq. 57, then

$$J_E = \sum_{r=1}^{m} B_r^2 + \sum_{r=m+1}^{\infty} B_r^2 \qquad (m = n) \tag{60a}$$

$$J_E = \sum_{r=1}^{n} f_r^2(B_{n+1}, \ldots, B_m) + \sum_{r=n+1}^{\infty} B_r^2 \qquad (m > n) \tag{60b}$$

The cost constraint (Eq. 60a for $m = n$) attains a minimum when

$$B_r = 0 \qquad (r = n + 1, \; n + 2, \ldots, \infty) \tag{61}$$

and the remaining n coefficients are solutions of Eqs. 57. Thus the optimum control is completely and uniquely determined.

In the case $m > n$, the cost constraint attains a minimum when

$$\sum_{r=1}^{n} f_r(B_{n+1}, \ldots, B_m) \frac{\partial}{\partial B_p} f_r(B_{n+1}, \ldots, B_m) + B_p = 0 \qquad (p = n + 1, \ldots, m) \tag{62}$$

$$B_r = 0 \qquad (r = m + 1, \; m + 2, \ldots, \infty) \tag{63}$$

Equations 57, 59, 62, and 63 provide the necessary m relations for the determination of the m coefficients of the optimum input control. If these coefficients are denoted by $(B^k_\mu)^*$, then the optimum control is

$$u_k^*(t) = \sum_{\mu=1}^{m^k} (B^k_\mu)^* \, \phi^k_\mu(t) \tag{64}$$

Special Case As a specific case of the preceding discussion, consider the system

$$y_1(t) = \xi_1 + \xi_2 t + \int_0^t (t - \sigma)\, u(\sigma)\, d\sigma \qquad (65)$$

$$y_2(t) = \xi_2 + \int_0^t u(\sigma)\, d\sigma \qquad (66)$$

Find the input that brings the state variables to $y_1 = y_2 = 0$ at $t = T$ and minimizes the integral

$$J_E = \int_0^T u^2(\sigma)\, d\sigma \qquad (67)$$

To this end, note that, at the terminal time,

$$-\xi_1 - T\,\xi_2 = \int_0^T (T - \sigma)\, u(\sigma)\, d\sigma \qquad (68)$$

$$-\xi_2 = \int_0^T u(\sigma)\, d\sigma \qquad (69)$$

The characteristic set of orthonormal functions is

$$\phi_1(t) = \left(\frac{1}{T}\right)^{1/2} \qquad \phi_2(t) = \left(\frac{3}{T}\right)^{1/2} - \left(\frac{12}{T^3}\right)^{1/2} t \qquad \ldots \qquad (70)$$

When the kernels of Eqs. 68 and 69 are expanded in terms of this set and the input is written in the form

$$u(\sigma) = \sum_{k=1}^{\infty} B_k\, \phi_k(\sigma) \qquad (71)$$

it is found that

$$B_1 = -\frac{\xi_2}{T^{1/2}} \qquad B_2 = -\left(\frac{12}{T^3}\right)^{1/2}\left(\xi_1 + \xi_2 \frac{T}{2}\right)$$

$$B_k = \text{arbitrary} \qquad (k \geq 3) \qquad (72)$$

Minimization of J_E (Eq. 67) results in $B_k = 0$ $(k \geq 3)$. Therefore the optimum input is

$$u^*(t) = -\frac{2}{T^2}(3\xi_1 + 2\xi_2 T) + \frac{6}{T^3}(2\xi_1 + \xi_2 T)t \tag{73}$$

A Nonlinear Optimum Control Problem

The procedure for nonlinear systems is similar to that developed for linear systems (see preceding section). Admittedly, however, the resulting algebraic equations for the expansion coefficients for the optimum control can be much more involved. For brevity, only a simple example will be discussed.

Consider the nonlinear system given by

$$y_1(t) = \int_0^t (t-\sigma)\, u(\sigma)\, d\sigma + \int_0^t \int_0^t (t-\sigma_1)(t-\sigma_2)\, u(\sigma_1)\, u(\sigma_2)\, d\sigma_1\, d\sigma_2 \tag{74}$$

$$y_2(t) = -2\int_0^t u(\sigma)\, d\sigma + \int_0^t \int_0^t u(\sigma_1)\, u(\sigma_2)\, d\sigma_1\, d\sigma_2 \tag{75}$$

Suppose that it is desired to find the input that brings the output to $y_1 = 1$ and $y_2 = -1$ at $t = 1$ and minimizes

$$J_E = \int_0^1 u^2(\sigma)\, d\sigma \tag{76}$$

To this end, note that the characteristic orthonormal set is the same as that of the special example of the preceding section for $T = 1$. Using the expansion technique, it is found that

$$B_1 = 1 \qquad B_2 = -1 \pm \sqrt{4 + \sqrt{12}} \tag{77}$$

Finally, minimization of J_E yields the optimum input as

$$u^*(t) = 1 + (-1 + \tfrac{1}{2}\sqrt{4 + \sqrt{12}})(\sqrt{3} - \sqrt{12}\, t) \tag{78}$$

In summary, it is felt that the input–output approach to nonlinear optimum control problems will yield many practical results.

SPACE–TIME DYNAMICS

General Remarks

It is well known that both the magnitude and the rapidity of flux shape changes, either in ordinary space or in velocity space, have a substantial influence on the dynamic behavior of nuclear reactors. Our efforts in this area center around the understanding of effects of such changes on a variety of measurements performed on reactors. More specifically, one of my students, Larry Foulke, has been investigating the interpretation of oscillation tests in the light of space–time effects; with S. O. Johnson, of Phillips Petroleum Co., and his group, we have also been interested in the interpretation of power excursion bursts. I will describe briefly these two efforts.

Interpretation of Oscillation Tests

Oscillation tests, or, in general, small perturbation tests, are performed to measure transfer functions either to design the reactor regulating system or to investigate stability (see also the discussion of A New Criterion of Asymptotic Stability). Mathematically the reactor transfer functions characterize the linear constant-coefficient approximation of reactor dynamics. Reactor transfer functions are by necessity related to the derived concept of reactivity, and they are functions of frequency only. Physically the reactor transfer functions describe the dynamic behavior of the undistorted fundamental flux mode. This behavior, however, may not be possible to achieve in a particular small perturbation experiment. For example, it is found experimentally that *neutron flux oscillations*, corresponding to a *localized oscillating absorber*, are functions of both the frequency of oscillation and the position of the detector.[10] This is particularly true at high frequencies, namely, rapid flux changes. The position dependence results because of the excitation of higher order modes or, equivalently, because of the finite time necessary for the propagation of disturbances.

Two related questions arise at this point: (1) Are the parameters of transfer functions space dependent? (2) How does one extract the parameters of normal transfer functions from oscillation tests?

The answer to the first question is no, despite statements to the contrary which have appeared in the literature.[10,11] The reason is that the same high modes that contribute to the dependence of the flux oscillations on position contribute also to the reactivity that corresponds to the oscillating absorber. When reactivity is computed in a consistent manner, then the ratio of flux to reactivity is independent of position, and a normal, space-independent transfer function is recovered. In other words, for a given detector position, the reactivity corresponding to an oscillating absorber differs from the latter in amplitude and phase by as much as the amplitude and phase of the detected neutron flux oscillations differ from the expected space-independent amplitude and phase results. This point is illustrated in Fig. 8.

The second question can be answered, in principle, as follows: Flux oscillations must be measured simultaneously at more than one position, and the results of the measurements must be analyzed in terms of a number of appropriate eigenfunctions to establish the higher

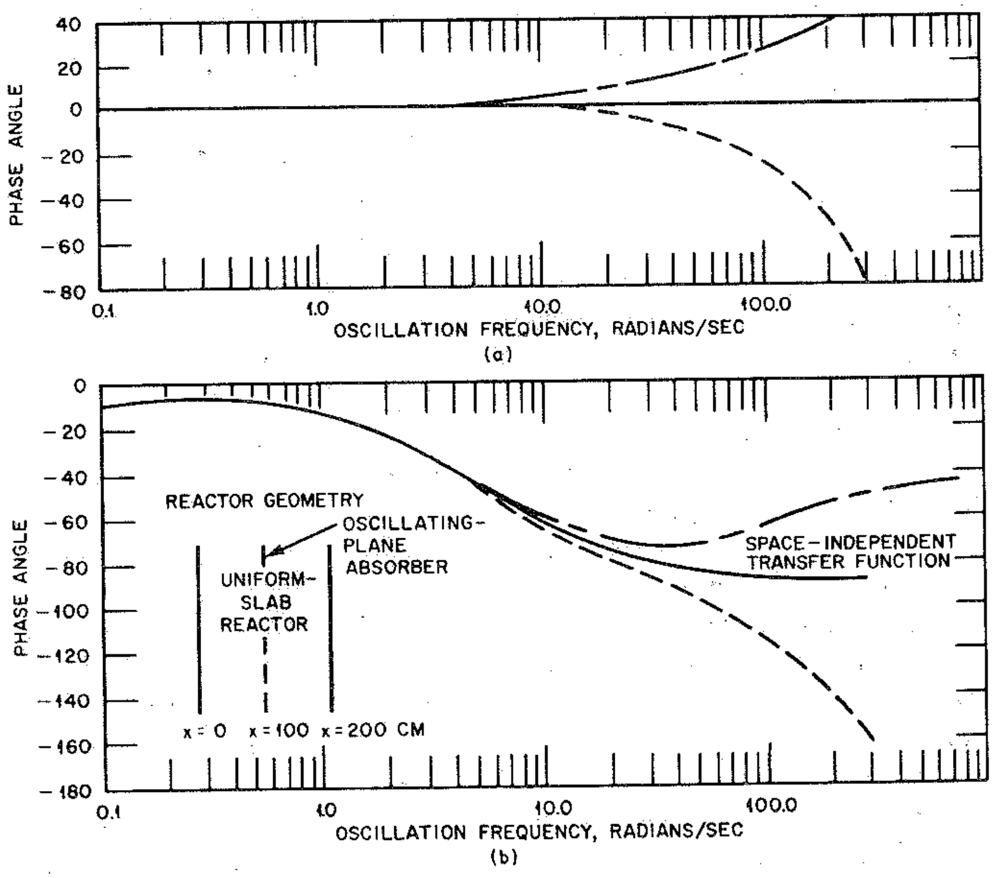


Fig. 8 – Computed phase angles (a) between oscillating-plane absorber and ρ/Λ corresponding to different detector positions and (b) between oscillating-plane absorber and neutron oscillations at different detector positions. The reactor geometry is shown in (b). Note that the departure of the phase of ρ/Λ from 0° is identical to the corresponding departure of the phase of the neutron oscillations from that of the space-independent transfer function. Similar results have been computed for the amplitudes. — - — shows the detector at x = 100 cm; ----- shows the detector at x = 10 cm.

mode distortion. Then reactivity can be computed, and transfer functions can be derived as described previously.

Finally, I would like to mention that the distortion at high frequencies, due to spatial modes, is relevant to the interpretation of recent techniques for the measurement of subcriticality. It seems to me that the spatial effects have not been duly considered in these subcriticality measurements.

Correlation of Power-excursion Bursts

Power-excursion bursts are often correlated by means of point reactor kinetics to establish inherent feedback coefficients and safety margins of nuclear reactors. Because of spatial dynamic effects, particular care must be exercised whenever such correlations are attempted.

To illustrate this point, some numerical results derived by means of point reactor kinetics and by means of space–time computations are presented in Refs. 5 and 12. The reactors considered in these references are excited by a step-ramp change of the fission cross sections in one region of the reactor. No feedback is included in the computations. Comparison of the results reveals that serious discrepancies can arise between the two types of calculations (see Fig. 9).

Currently S. O. Johnson and I are trying to investigate the same problem for reactors with a variety of inherent feedback mechanisms. The results will be reported[13] at the 1965 American Nuclear Society (ANS) meeting in Gatlinburg, Tenn. For purposes of this discussion, it suffices to say that discrepancies between space–time and point kinetics computations of power-excursion bursts do exist even in the presence of feedback. These discrepancies are experienced either in the shape of the power burst or in the magnitude of the average power. Therefore special care must be exercised whenever power-excursion bursts are correlated by means of point kinetics because the burst shape is indicative of the energy dependence of the feedback mechanism and because the integral under the power burst is related to the safety margin of the reactor.

DYNAMIC PROGRAMMING

Professor Henri Fenech and Dr. Ian Wall have used dynamic programming to investigate the optimum refueling policy for a power nuclear reactor. Even though the authors have presented their findings at previous ANS meetings, I thought it might be useful if I brought their work

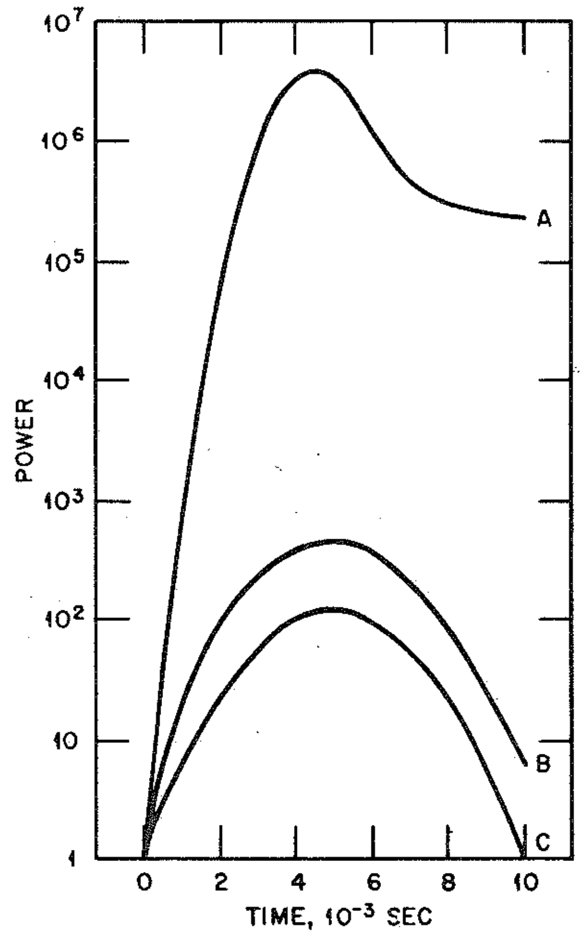


Fig. 9 — Curve A is the computed average power for a 240-cm slab; curve B is the computed average power for a 60-cm slab computed by means of a space–time code; curve C indicates the average power that would have been derived by means of point kinetics for both reactors (see Refs. 5 and 12).

to your attention because their papers have usually been scheduled in fuel-cycle sessions.

Ordinarily refueling policies are based either on 100% batch irradiation or on discontinuous out–in refueling procedures. For a given reactor design, the out–in policy results in unit energy cost savings of the order of 0.2 to 0.5 mill/kw-hr over that of the 100% batch policy. Typical results are summarized in Table 1.

Fenech and Wall investigated the optimum refueling policy by examining which part of the fuel should be discharged and which part should be rearranged at every refueling step during the life of the plant. By means of the method of dynamic programming, they reduced the complexity of the problem to a level amenable to com-

Table 1 – SALIENT DETAILS OF TWO NONOPTIMIZED POLICIES

Policy	Power cost, mills/kw-hr	Maximum power peaking	Maximum burnup, Mwd/ton	Average burnup, Mwd/ton	Plant life, years	Number of stages
100% batch	6.512	1.86	17,880	14,880	30.34	16
Three-zone out–in	6.140	1.74	20,620	19,750	30.31	33

Table 2 – POWER COSTS OF OPTIMIZED POLICIES AS A FUNCTION OF MAXIMUM BURNUP, POWER PEAKING, AND REFUELING TIME

Maximum permissible burnup, Mwd/ton	Maximum permissible power peaking	Refueling time, days	Power costs, mills/kw-hr	Plant life, years	Number of stages	Average burnup, Mwd/ton
20,000	2.5	14	6.152	30.93	24	18,980
20,500	2.5	14	6.147	30.27	24	19,060
21,000	2.5	14	6.089	30.86	32	20,350
21,500	2.5	14	6.062	30.62	32	20,790
21,500	2.5	28	6.224	30.60	27	20,200
21,500	2.5	42	6.365	30.29	25	20,670
22,000	2.25	14	6.009	30.61	31	21,420
22,000	2.0	14	6.073	30.40	32	20,620
22,000	1.75	14	6.115	30.43	26	19,650
22,000	1.65	14	6.303	31.40	21	17,100
22,000	1.50	14	*	*	*	*

*No admissible policies.

putation. Some typical results of the investigation are given in Table 2. It is seen from the data presented in this table that an optimized policy results in further savings over those of the out–in policy. The detailed statement of the problem and a description of the necessary digital codes are given in Ref. 14.

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