ON THE MEASUREMENT OF CHARACTERISTIC KERNELS OF A CLASS OF NONLINEAR SYSTEMS

by

Richard J. Hooper* and Elias P. Gyftopoulos

Department of Nuclear Engineering **
and Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Presented at the Symposium on Neutron Noise, Waves and Pulse Propagation, held at the University of Florida, Gainesville, Florida, February 14-16, 1966.

---

* Present Address: Knolls Atomic Power Laboratory
Schenectady, New York

** This work was partially supported by the National Science Foundation (Grant GK-524)

I. INTRODUCTION

The purpose of this paper is to present the results of some computer "experiments" in which the dynamic response characteristics of certain nonlinear systems are measured by means of input-output cross correlation techniques. Ternary periodic signals are used as input signals. The use of such inputs allows a substantial reduction in the crosscorrelation time required to obtain accurate results compared to that needed when the input is Gaussian white noise. In addition, ternary periodic signals are better suited for the measurement of the linear approximation characteristics, in the presence of nonlinearities, than binary periodic signals.

Wiener(1) has shown that the output of a wide class of nonlinear systems can be expressed as an infinite sum of orthogonal, inhomogeneous functionals of the input, when the latter is the stochastic function. The kernels of the Wiener functionals characterize a nonlinear system in the same way that the impulse response characterizes a linear system.

Schetzen and Lee(2) have shown that the kernels of the Wiener functionals can be determined experimentally by means of input-output crosscorrelation techniques, when the input is Gaussian white noise. Such experiments, however, require very long crosscorrelation times and may be impractical for some applications.

Gyftopoulos[2] suggested that this impracticality might be avoided by designing periodic input signals whose correlation functions, over a finite time interval and up to some order K, are approximately equal to those of Gaussian white noise. The ternary periodic signals used in this paper have first, second and third order correlation functions which satisfy this requirement. Consequently, they are suitable for the measurement of the kernels of nonlinear systems which can be characterized by only the first three Wiener functionals. Equivalently, these signals are suitable for the measurement of the kernels $h_1(\tau)$ and $h_2(\tau_1,\tau_2)$ of nonlinear systems represented by the input-output Volterra relation:

$$y(t) = \int_0^t h_1(\tau)x(t-\tau)d\tau + \int_0^t\int_0^t h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \tag{1}$$

where $x(t)$ is the input and $y(t)$ is the output. As discussed in reference 4, the ternary periodic signals are also suitable for the measurement of the kernel $h_1(\tau)$ of a broader class of systems than described by Eq. 1.

What follows is a brief description of the ternary periodic signals, the measurement procedures, and the results of measurements performed on several computer simulated systems.

II. TERNARY PERIODIC SIGNALS

The properties of the ternary periodic signals used in this work are discussed elsewhere[4]. Some of these properties, however, are repeated here for convenience.

Synthesis procedure. The signals are constructed in the following manner:

1. At any instant of time, the signal may assume only one of three possible (normalized) values: +1, 0, -1.
2. The signal is discontinuous and may change value only at event points having uniform spacing Δt.
3. The signal is periodic with period $T=N\Delta t$, where $N=3^n-1$, n=integer.
4. The j-th bit C_j of the signal is generated by the linear recursion formula:

$$C_j = a_1C_{j-1} + a_2C_{j-2} + \cdots a_nC_{j-n} \quad , \qquad (2)$$

where the right-hand side of Eq. 2 is reduced modulo-3 so that C_j satisfies rule 1. The coefficient a_i is an integer restricted to either of the three values +1, 0, -1 or, equivalently, +2, +1, 0. The sets of coefficients $a_1, a_2, \cdots a_n$ which are known to generate sequences of length 3^n-1, for $2 < n \leq 8$, are given in reference 5. Some of these sets are listed in Table 1. A typical ternary periodic signal, for $n=3$, $N=26$, $a_1=1$, $a_2=a_3=2$, is shown in Fig. 1.

Correlation Functions. The k-th order correlation function of a periodic signal $x(t)$, of period T, is a periodic function of the same period defined by the relation:

$$\phi_k(\tau_1,\tau_2,\cdots\tau_k) = \frac{1}{T}\int_0^T x(t)x(t+\tau_1)\cdots x(t+\tau_k)dt \quad . \qquad (3)$$

TABLE [illegible]

Some Typical Recursion [illegible]

for Ternary Chains of Length [illegible]

n	N	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
3	26	1	2	2					
3	26	1	0	2					
4	80	2	1	1	1				
4	80	1	0	0	1				
5	242	0	2	1	1	2			
5	242	2	0	2	2	2			
6	728	1	1	1	0	1	1		
6	728	1	0	1	0	0	1		
7	2186	2	1	1	1	1	1	2	
7	2186	2	1	1	2	2	1	2	
8	6560	2	1	2	1	2	1	1	1
8	6560	0	0	1	0	1	2	1	1

For the ternary periodic signals it is found that

1. The first order correlation function is given by the relation(4):

$$\phi_1(\tau) \cong \frac{2T}{3^{n+1}} \sum_{-\infty}^{\infty} (-1)^p \delta(\tau - \frac{pT}{2}) \qquad \text{for } N \gg 1 \;, \qquad (4)$$

where $\delta(\tau)$ is the delta function. It is clear that $\phi_1(\tau)$ over the interval $0 < \tau < T/2$ is approximately equal to the corresponding correlation function of Gaussian white noise.

2. The even order correlation functions are identically equal to zero, namely:

$$\phi_{2\ell}(\tau_1, \tau_2, \ldots \tau_{2\ell}) \equiv 0 \qquad \text{for all } \ell \geq 1 \;. \qquad (5)$$

These correlations are identically equal to the corresponding correlations of Gaussian white noise.

3. Over one period, the third order correlation function $\phi_3(\tau_1, \tau_2, \tau_3)$ assumes its maximum value at $\tau_1 = \tau_2 = \tau_3 = 0$ and has the property:

$$\phi_3(0, \tau, \tau) = \frac{2}{3}\phi_3(0,0,0) \qquad \text{for } \tau \neq 0 \;. \qquad (6)$$

In addition, throughout the $(\tau_1 \neq \tau_2 \neq \tau_3)$-space, $\phi_3(\tau_1, \tau_2, \tau_3)$ is identically equal to zero except for a relatively small number of "anomalous" regions, of size ΔT^3, where it varies from zero to either of the values $\pm\phi_3(0,0,0)/3$. These "anomalous" regions are fairly uniformly distributed in the $(\tau_1 \neq \tau_2 \neq \tau_3)$-space, although an analytical method for predicting their exact location is not presently available. Examples of

[illegible]

an approximation to the [illegible]

Gaussian white noise. Nevertheless, [illegible]

for the purposes of this paper as discussed in the following sections.

III. MEASUREMENT PROCEDURES

In the system described by Eq. 1, the kernel $h_1(\tau)$ can be measured by exciting the system by means of a ternary periodic input and crosscorrelating input and output in the manner depicted in Fig. 2. Indeed, the output of the integrator after one period is given by the relation:

$$\int_0^T x(t-\tau)y(t)dt = T\int_0^t h_1(\lambda)\phi_2(\tau-\lambda)d\lambda \quad . \tag{7}$$

since the second order correlation function of the input is identically equal to zero. If the period T of the ternary signal is chosen at least twice as long as the settling time T_{st} of the kernel $h_1(\tau)$,* then use of Eq. 4 in Eq. 7 yields:

$$h_1(\tau) \cong \frac{1}{2xg^{n-1}(\Delta t)^{2}}\int_0^T x(t-\tau)y(t)dt \quad . \tag{8}$$

Regarding the kernel $h_2(\tau_1,\tau_2)$, if a ternary periodic input is used in the crosscorrelator shown in Fig. 3, then the output of the integrator after one period is given by

---

* The settling time is defined as the time beyond which the normalized, absolute [illegible] of the kernel is smaller than ϵ, [illegible] positive number, arbitrarily [illegible].

the relation:

$$\int_0^T x(t-\tau_1)x(t-\tau_2)y(t)dt =$$

$$= T\int_0^t\int_0^t h_2(\lambda_1,\lambda_2)R_3(\lambda_1-\lambda_2,\lambda_1-\tau_1,\lambda_1-\tau_2)d\lambda_1 d\lambda_2 \qquad (9)$$

since the second order correlation function of the input is identically equal to zero. Use of the properties of $R_3(\tau_1,\tau_2,\tau_3)$ in Eq. 9 for $\tau_1 \neq \tau_2$ yields:

$$\int_0^T x(t-\tau_1)x(t-\tau_2)y(t)dt =$$

$$= 8x3^{n-2}(\Delta\tau)^3 h_2(\tau_1,\tau_2) + \left\{\begin{matrix}\text{Contribution} \\ \text{from "anomalous"} \\ \text{regions}\end{matrix}\right\} \text{ for } \tau_1 \neq \tau_2 \qquad (10)$$

If the third order correlation function were zero in the "anomalous" regions, then Eq. 10 could be written as:

$$h_2(\tau_1,\tau_2) = \frac{1}{8x3^{n-2}(\Delta\tau)^3}\int_0^T x(t-\tau_1)x(t-\tau_2)y(t)dt \quad \text{for } \tau_1 \neq \tau_2 \qquad (11)$$

Efforts to modify the ternary signals in a way that would eliminate altogether the "anomalous" regions proved unsuccessful. Since no other families of signals are presently available, whose correlation functions up to third order are better approximations to those of Gaussian white noise, it behooves us to find whether, through proper choice of the period T, Eq. 11 is a good approximation for $h_2(\tau_1,\tau_2)$ in spite of the "anomalous" regions. This question cannot be answered analytically because there is no known way of predicting [illegible] of the "anomalous" regions. [illegible] performing [illegible]

[illegible]

done in the next section. [illegible] found that Eq. 11 is approximately [illegible] hundred times the settling time T_{s2} of $h_2(\tau_1,\tau_2)$. This result is to be expected. By choosing $T \gg T_{s2}$, the values of $K_3(\tau_1,\tau_2,\tau_3)$ in the "anomalous" regions occur where $h_2(\tau_1,\tau_2)$ is negligibly small. Therefore, these values do not contribute appreciably to the integral in the right-hand side of Eq. 9.

IV. "EXPERIMENTAL" RESULTS

General Remarks. Several simulated experiments are carried out in which the kernels $h_1(\tau)$ and $h_2(\tau_1,\tau_2)$ of known systems are measured by means of the procedures depicted in Figs. 2 and 3, respectively. The experiments are simulated in the sense that instead of synthesizing a system according to Eq. 1 and of measuring the output to a particular ternary input, this is done by means of a digital computer. Once the output is calculated, the computer proceeds with the evaluation of the integrals on the right-hand side of Eqs. 8 and 11. The integrals are in turn compared to the known $h_1(\tau)$ and $h_2(\tau_1,\tau_2)$, respectively. Note that these computations do not involve explicitly any assumptions about any properties of the ternary input.

Excellent agreement between the computed and the known [illegible] when [illegible] T_{s1}. On the other hand, [illegible] agreement, it is necessary that T be about $100T_{s2}$. Shorter crosscorrelation times result in appreciable errors, the errors being larger the shorter the period T.

The settling time for all the kernels is arbitrarily taken equal to seven times the characteristic decay constant of the kernel.

Results. The first system investigated has the kernels:

$$h_1(\tau) = \exp(-\tau) \quad \text{and} \quad h_2(\tau_1,\tau_2) = \exp(-\tau_1-\tau_2) \quad . \tag{12}$$

The kernel $h_1(\tau)$ is measured using a ternary periodic input with $N=80$, $\Delta T=0.173$, $T=14.0$, and a ratio $T/T_{s1}=2$. The results are shown in Fig. 4. The solid curve is the known kernel and the circles are the measured values. The agreement is excellent. The mean of the absolute value of the percentage difference between measured and respective expected values is 0.16%.

When the kernel $h_2(\tau_1,\tau_2)$ is measured using the same input as above, the results are so badly scattered as to be meaningless. When, however, the input is taken much longer, namely with $N=2186$, $\Delta T=0.269$ and $T=595$, corresponding to a ratio $T/T_{s2}=85$, some typical measured values of $h_2(\tau_1,\tau_2)$ are shown in Figs. 5 and 6. These values are representative of results obtained throughout the (τ_1,τ_2)-plane. [illegible]

[illegible]

For a=0.6667 and ω=[illegible].0974, the ternary input used to measure $h_2(\tau_1,\tau_2)$ has N=6560, Δf=0.15 and T=98.4. The ratio of the period over the settling time (T/a) is 94. Typical results of this measurement are shown in Figs. 7 and 8. The root-mean-square, σ_{rms}, of the difference between measured and expected values, expressed as a percentage of the peak value of $h_2(\tau_1,\tau_2)$, has the values 2.1% and 0.57% for the data of Figs. 7 and 8, respectively.

Similar results are found when a=2 and ω=2.09 and the ternary input has N=2186, Δf=0.14, T=305 and T/T_{s2}=87.

Comparison with other Results. Previous to the work described here, Widnall(6) used a Gaussian white noise input to measure the kernel $h_2(\tau_1,\tau_2)$ of a system similar to that defined by Eq. 13 and for $a \approx 0.6$ and $\omega \approx 2.13$. Thus, an interesting comparison of the results obtainable by the two types of inputs can be made.

The accuracy of the results obtained in reference 6 is estimated by computing the value of the parameter σ_{rms} for the data corresponding to $h_2(\tau_1,\tau_2^*)$, where τ_2^* is such that $h_2(\tau_1,\tau_2^*)$ is zero for all τ_1. When the length of the input-output records used in the crosscorrelator is about 7000 times the settling time T_{s2}, σ_{rms} has the value 0.6%. When the length of the input-output records is about 350 times the settling time, σ_{rms} has the value of 2.8%.

In the present study, using the ternary periodic input with a period 94 times the settling time, the σ_{rms} values, for three different values of τ_2^*, are 0.4%, 2.1%

and 0.4%, respectively, with an average of about 1.1%. Consequently, in this example, the only opportunity available for comparison, the ternary periodic input yields the kernel $h_2(\tau_1,\tau_2)$ with an accuracy comparable to that obtained using Gaussian white noise but with considerably less input-output data.

V. CONCLUSIONS

A family of discrete level, periodic ternary signals is proposed for the measurement of the kernels of a class of nonlinear systems. These systems are representable by the sum of a first and a second order functional of the input. The measurements are achieved by means of cross-correlations.

The ternary periodic signals allow a substantial reduction in the crosscorrelation time compared to that needed when the input is Gaussian white noise. In addition, these signals are better suited for the measurement of the first order kernel, in the presence of nonlinearities, than binary periodic signals. The reason is that the correlation functions of the latter, of order higher than the first, are very poor approximations to the corresponding functions of the Gaussian white noise.

Discrete, periodic signals are very useful for crosscorrelation measurements as discussed elsewhere(7)

REFERENCES

1. N. Wiener, "Nonlinear Problems in Random Theory", Technology Press-Wiley, 1958.

2. M. Schetzen, Y.W. Lee, "Measurement of the Kernels of a Nonlinear System by Crosscorrelation", Research Laboratory of Electronics (M.I.T.) Quarterly Progress Report, January, 1961.

3. E.P. Gyftopoulos, "On the Measurement Dynamic Characteristics of Nuclear Reactor Systems", Trans. Amer. Nuclear Soc., 4, 2 (Nov., 1961), p. 206.

4. E.P. Gyftopoulos, R.J. Hooper, "Signals for Transfer Function Measurements in Nonlinear Systems", [illegible] 7679, June, 1964.

5. R.J. Hooper, "Test Signals for a Class of Nonlinear Systems", ScD Thesis in Nuclear Engineering, Massachusetts Institute of Technology, 1966.

6. W.S. Widnall, "Measurement of the Second Degree Wiener Kernel in a Nonlinear System by Crosscorrelation", S.M. Thesis in Electrical Engineering, Massachusetts Institute of Technology, 1962

7. J.D. Balcomb, H.B. Demuth and E.P. Gyftopoulos, "A Crosscorrelation Method for Measuring the Impulse Response of Reactor Systems", Nucl. Sci. Eng., 11, 159-166 (1961).

LIST OF FIGURES

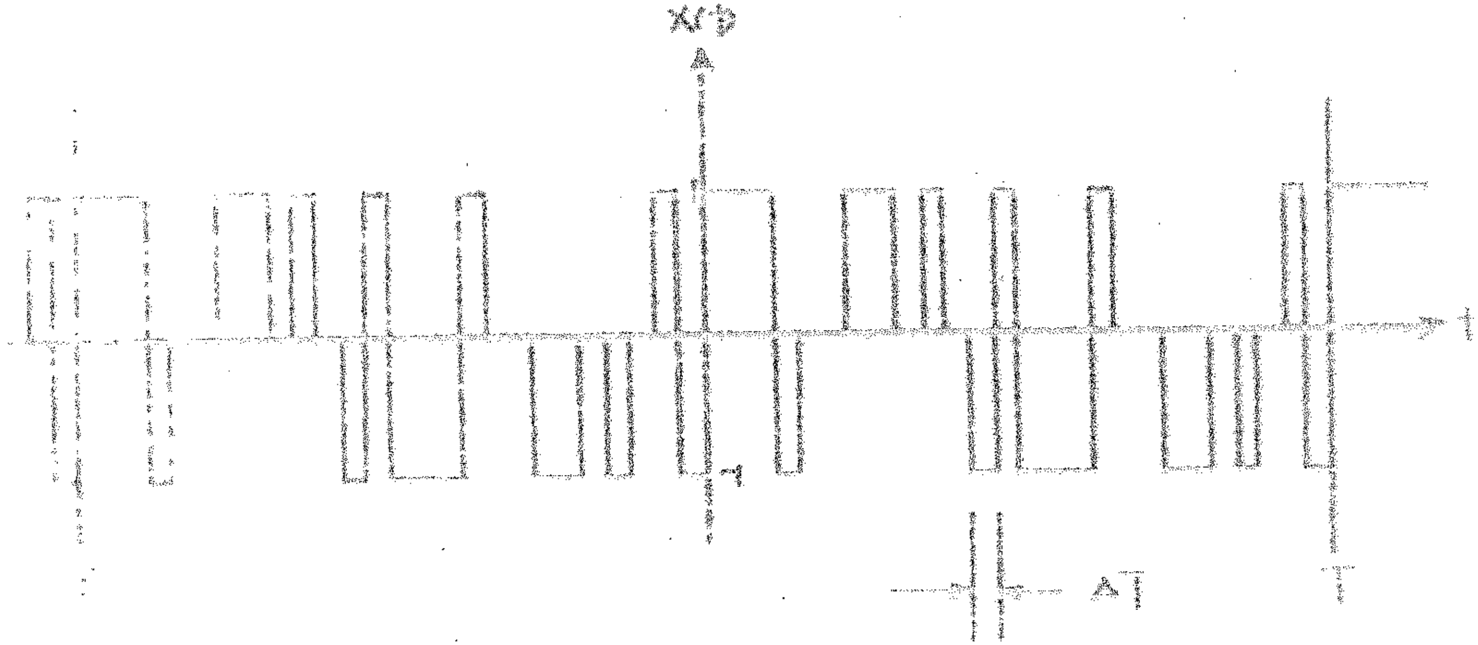
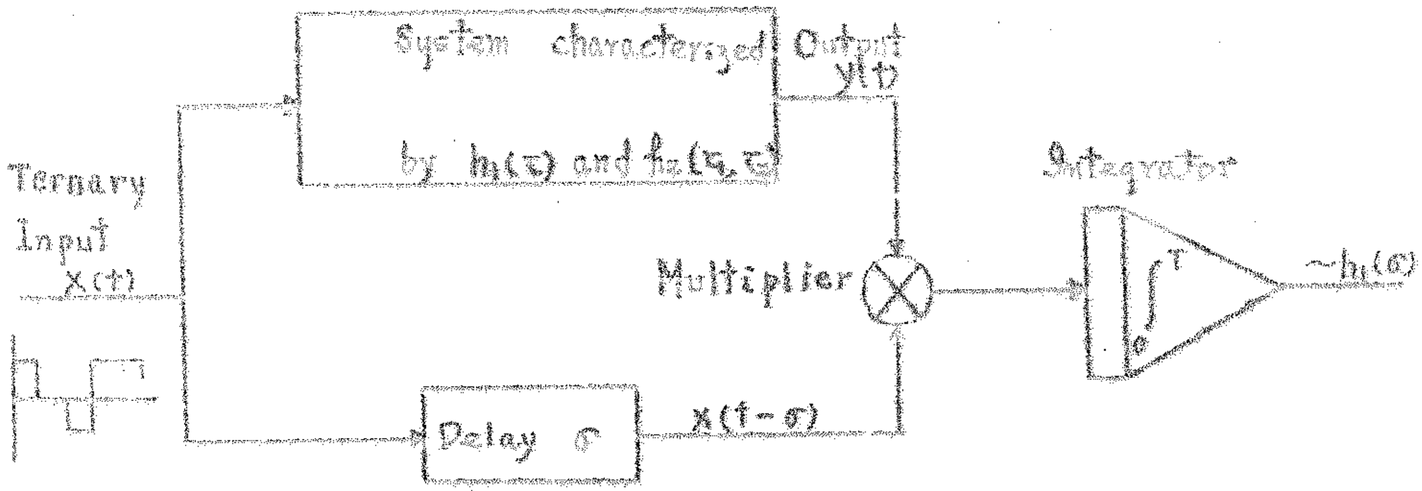


Fig. 1



Ternary
Input
X(t)
System characterized
by h1(τ) and h2(τ1, τ2)
Output
y(t)
Delay σ
X(t − σ)
Multiplier
Integrator
∫₀ᵀ
~h1(σ)

Fig. 2

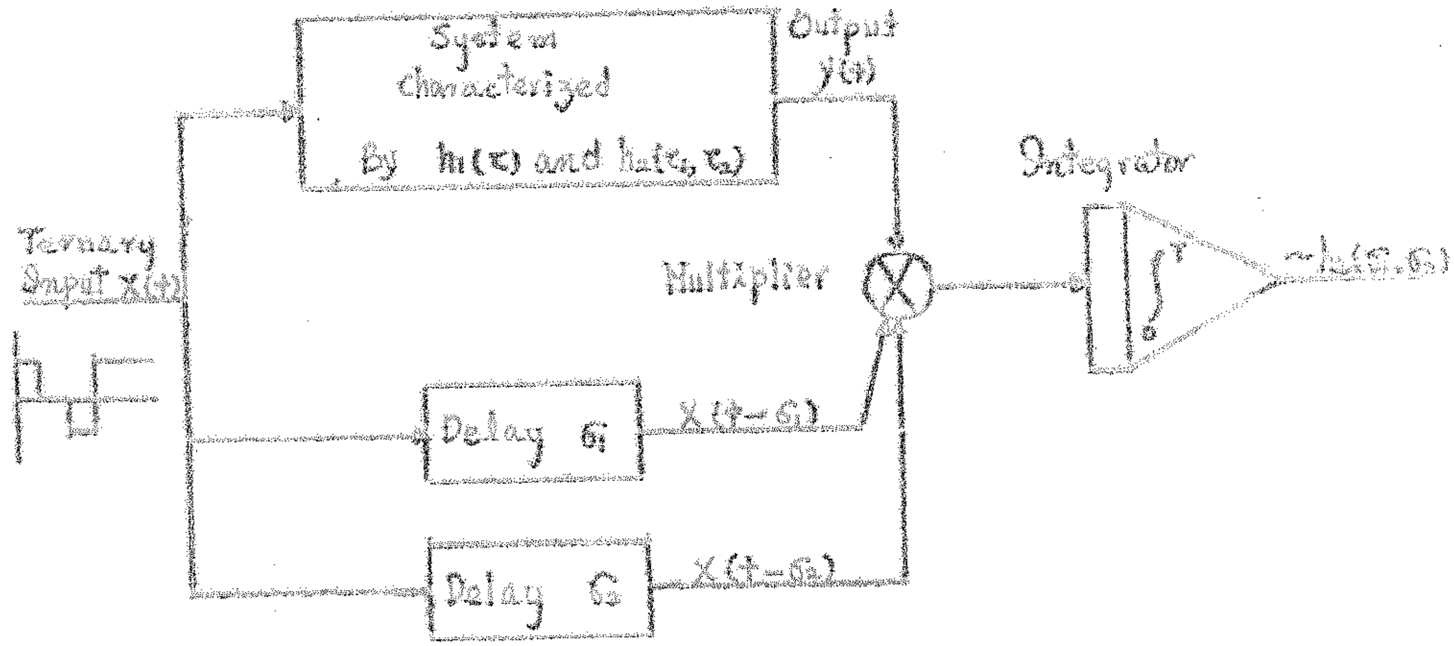

System
characterized
By $h_1(\tau)$ and $h_2(\tau_1, \tau_2)$
Output
$y(t)$
Ternary
Input $X(t)$
Multiplier
Integrator
$\sim h_2(\sigma_1, \sigma_2)$
Delay σ_1
$X(t-\sigma_1)$
Delay σ_2
$X(t-\sigma_2)$

Fig. 3

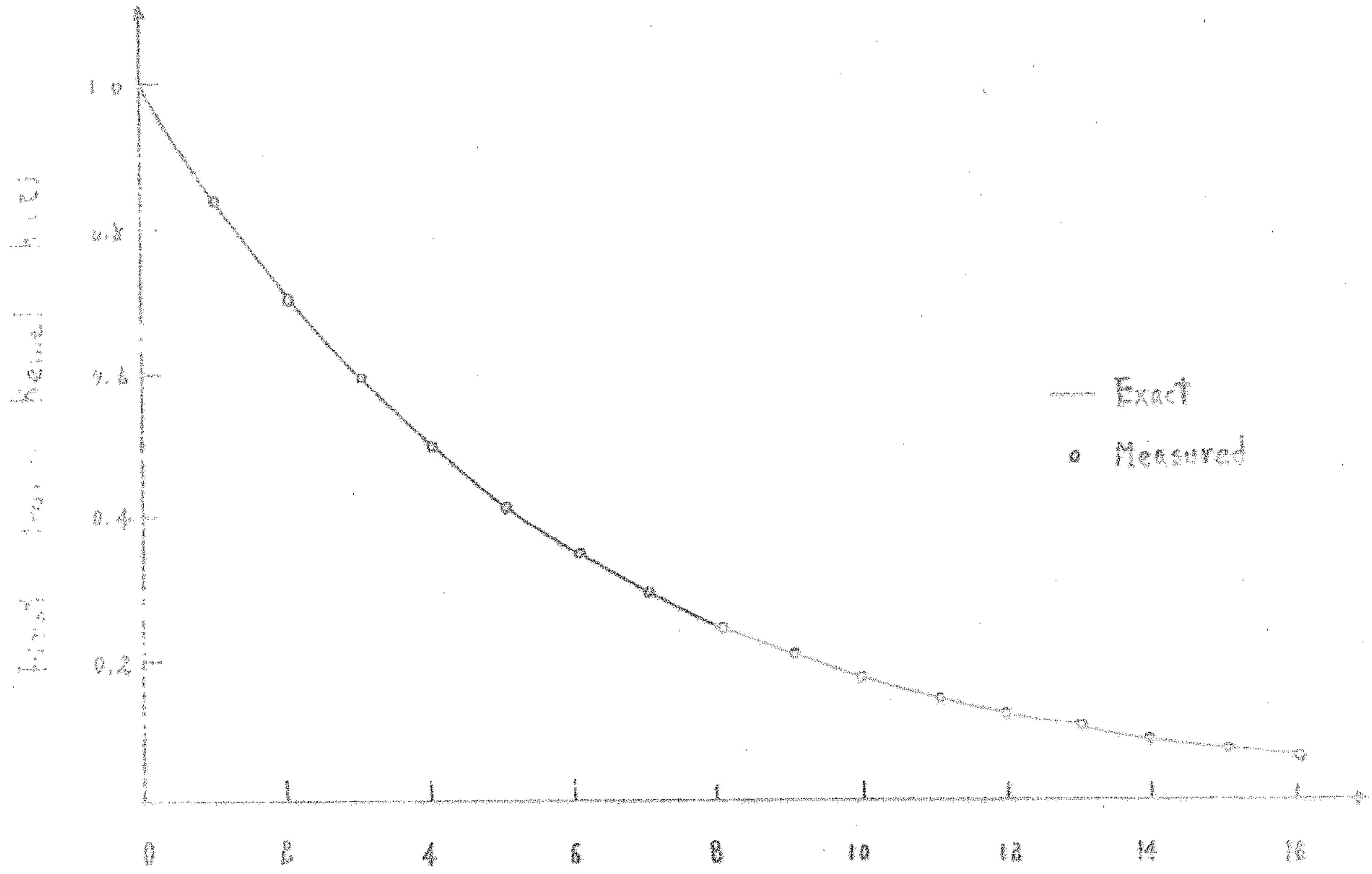
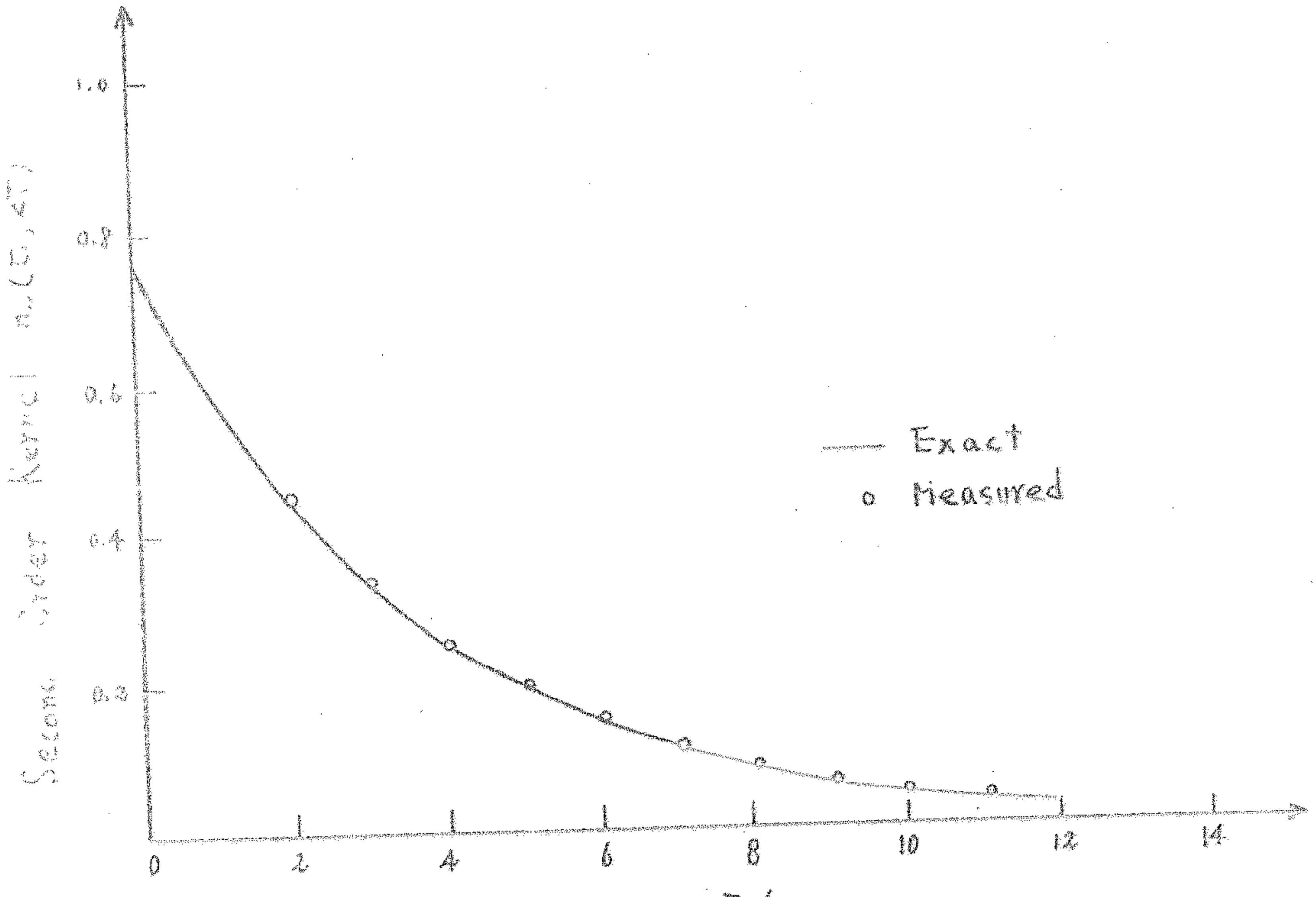


Fig. 4



Second Order Kernel
1.0
0.8
0.6
0.4
0.2
Exact
Measured
0
2
4
6
8
10
12
14

Fig. 5

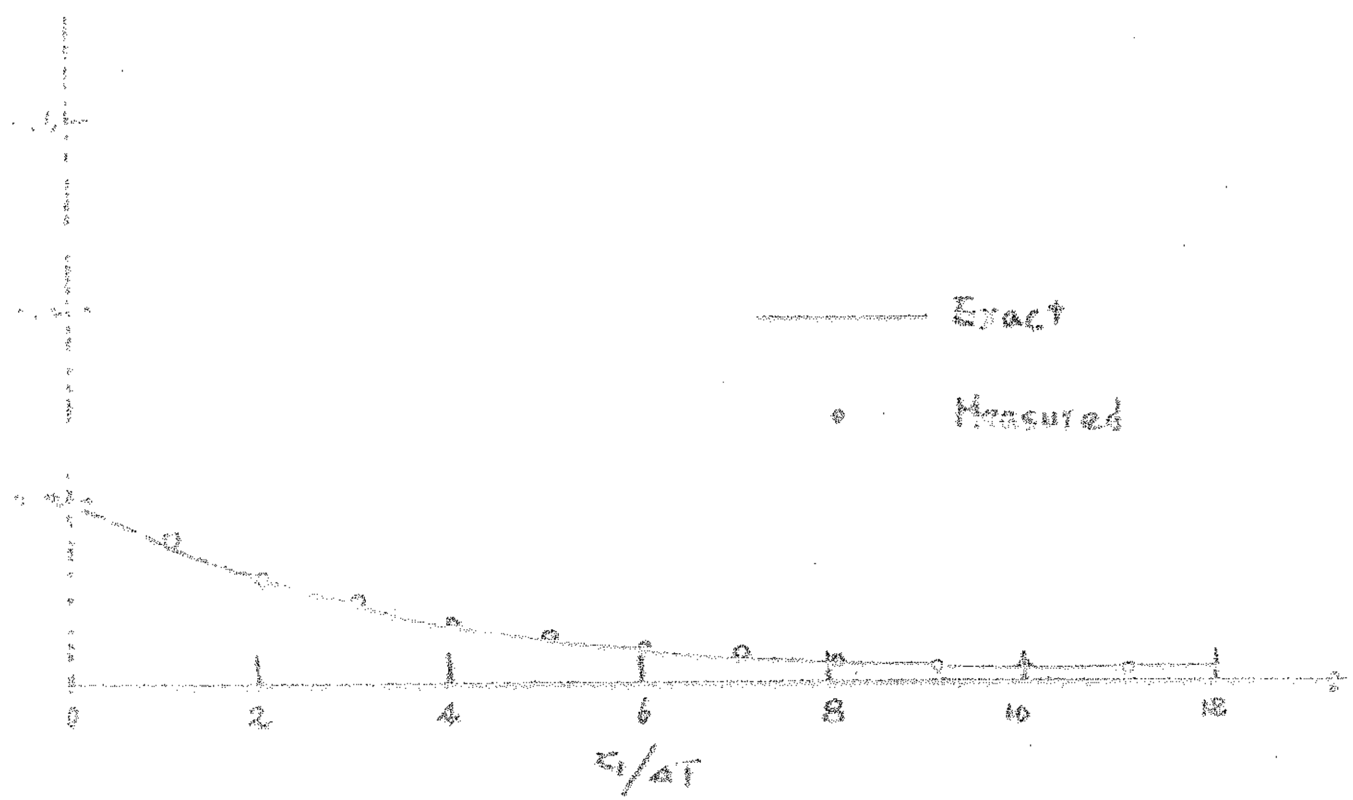


Fig. 6

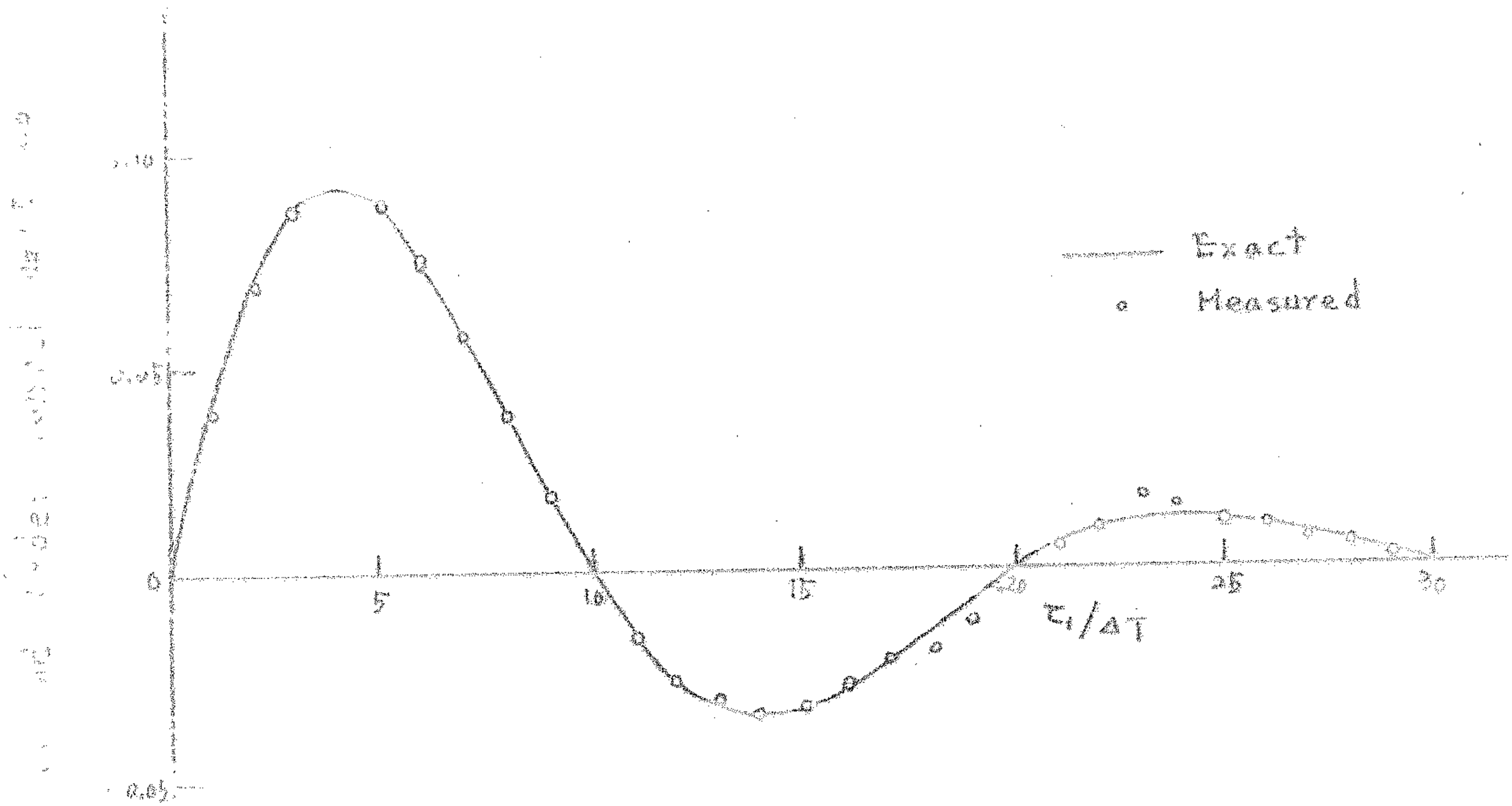


Fig. 7

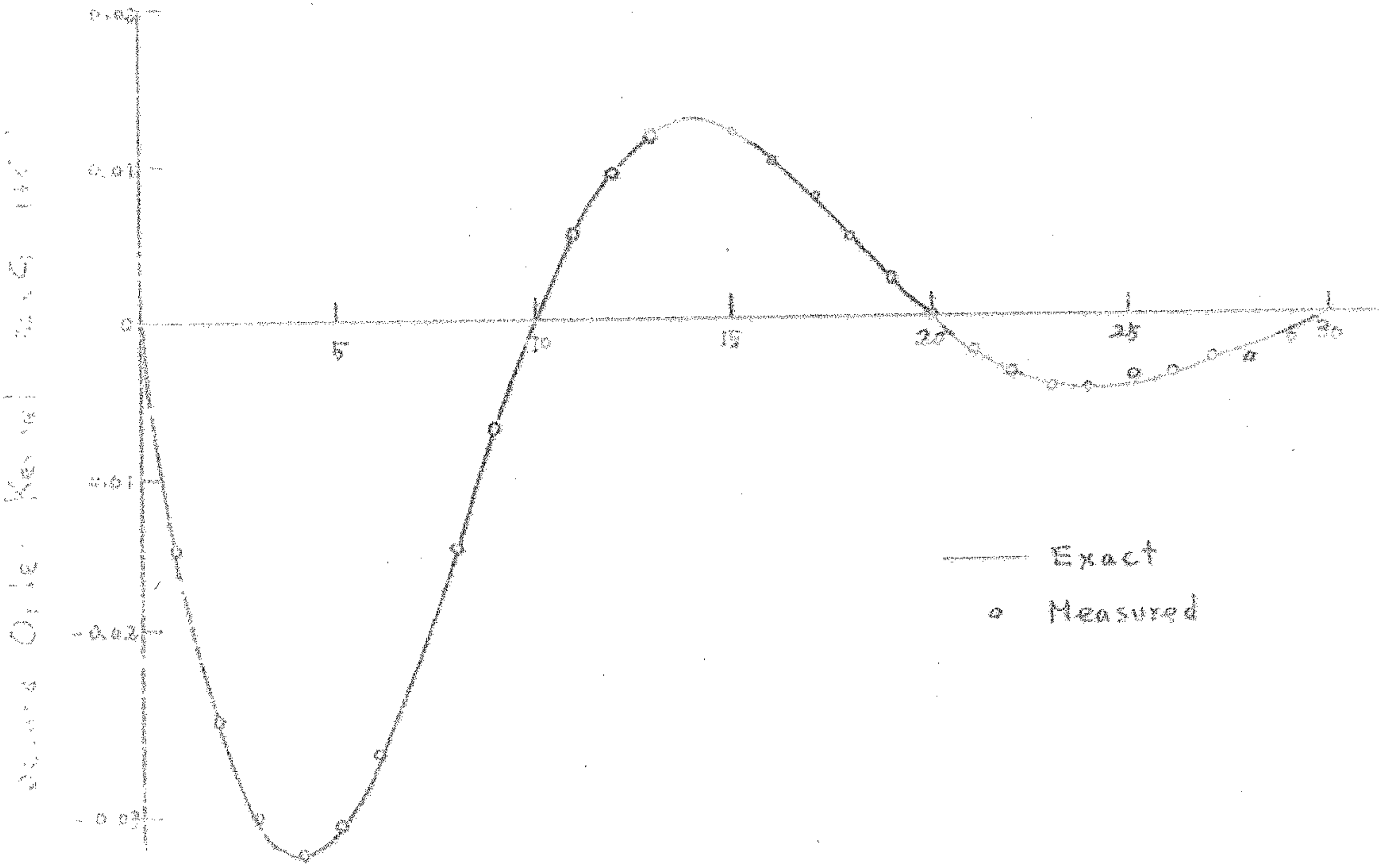


Fig. 8