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ON THE RANGE OF VALIDITY OF LINEARIZED BOLTZMANN EQUATIONS

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INTRODUCTION

The purpose of this paper is to obtain thermodynamically rigorous conditions for and the range of validity of the linearized Boltzmann equations used in analyses of the cesium vapor in the interelectrode space of a thermionic converter.

To this end, the conditions are expressed in terms of gradients of temperatures and total thermodynamic potentials only. Then it is shown that the linearized equations are valid in the plasma region for all practical values of electron flux, but become invalid in each of the sheath regions when the electron flux exceeds about 1/100 of the corresponding random flux. These results prove: (1) that the conditions for the validity of the linearized Boltzmann equations depend only on the gradients of the thermodynamic potentials and not on the gradients of pressure, temperature, and motive (electric potential); (2) that the range of validity is determined by the magnitudes of the particle fluxes; and (3) that the linearized Boltzmann equations and the corresponding hydrodynamic equations and fluxes can be used even in regions where the electric field is appreciable, such as a transition region between a steep sheath and a plasma.

LINEARIZATION OF BOLTZMANN EQUATIONS

As discussed by Wilkins (1966), the steady-state Boltzmann equations for the distribution functions of the various types of particles in a cesium vapor system, such as that in the interelectrode space of a thermionic converter, can be linearized and solved subject to four restrictions. The first restriction is that the distribution functions f_e for electrons, f_i for singly-charged cesium positive ions, and f_a for cesium atoms can be approximated by the expressions

$$f_e = f_{eo}[1 + \tilde{\phi}_e(x,\underline{v}_e)]; \quad f_i = f_{io}[1 + \tilde{\phi}_i(x,\underline{v}_i)];$$
$$\text{and } f_a = f_{ao}, \tag{1}$$

where, for α = e, i, and a,

$$f_{\alpha o} = n_\alpha \left(\frac{m_\alpha}{2\pi kT_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha v_\alpha^2}{2kT_\alpha}\right) \tag{2}$$

$$n_\alpha = \int f_\alpha d^3\underline{v}_\alpha; \quad 3n_\alpha kT_\alpha = \int m_\alpha v_\alpha^2 f_\alpha d^3\underline{v}_\alpha;$$
$$p_\alpha = n_\alpha kT_\alpha , \tag{3}$$

and, for α = e and i, $\tilde{\phi}_\alpha(x,\underline{v}_\alpha)$ is a slowly varying function of position x and velocity $\underline{v}_\alpha$ such that

$$|\tilde{\phi}_\alpha(x,\underline{v}_\alpha)| << 1. \tag{4}$$

We see from relations 3 that the particle density n_α, temperature T_α, and pressure p_α are functions of position x. For α = e and i, they are characteristic of the corresponding stable equilibrium state of the α-particle phase at position x.

The other three restrictions given by Wilkins (1966) refer to approximate expressions for the collision integrals and the equality of T_i and T_a.

When electron-electron elastic collisions are negligible compared with other types of collisions, then the linearized Boltzmann equations can be solved for the perturbation function $\tilde{\phi}_e(x,\underline{v}_e)$. The solution can be expressed in terms of thermodynamic potentials and is given by the relation

$$\tilde{\phi}_e(x,\underline{v}_e) = -\frac{v_{ex}}{kT_e\nu_e(v_e)}\left[\left(\frac{m_e v_e^2}{2} - \mu_{ce}\right)\frac{1}{T_e}\frac{dT_e}{dx} + \frac{d\mu_e}{dx}\right], \tag{5}$$

where v_{ex} denotes the component of $\underline{v}_e$ in the x-direction, $\nu_e(v_e)$ the electron collision frequency defined by the relation

$$\nu_e(v_e) = \nu_{ei}(v_e) + \nu_{ea}(v_e), \tag{6}$$

namely as the sum of the electron-ion $\nu_{ei}(v_e)$ and the electron-atom $\nu_{ea}(v_e)$ collision frequencies, m_e the electron mass, μ_e the electron total potential, namely

$$\mu_e = -kT_e \ln \frac{2(2\pi m_e)^{3/2}(kT_e)^{5/2}}{p_e h^3} + \psi$$
$$= \mu_{ce} + \psi , \tag{7}$$

h Planck's constant, μ_{ce} the electron chemical potential, and ψ the electron motive. We can readily verify that $\tilde{\phi}_e(x,\underline{v}_e)$ depends only on the gradients of $1/T_e$ and μ_e/T_e and not on the gradients of p_e, T_e, and ψ. These dependencies are consistent with the well known requirements of irreversible thermodynamics.

We see from Eq. 5 that $|\tilde{\phi}_e(x,\underline{v}_e)|$ will be much smaller than unity if and only if

$$\left|\frac{v_{ex}}{kT_e\,\nu_e(v_e)}\left(\frac{m_e v_e^2}{2} - \mu_{ce}\right)\frac{1}{T_e}\frac{dT_e}{dx}\right| << 1, \tag{8}$$

and

$$\left| \frac{v_{ex}}{kT_e \nu_e(v_e)} \frac{d\mu_e}{dx} \right| << 1. \tag{9}$$

For a given collision frequency, Eqs. 8 and 9 indicate that the limitations on the gradients dT_e/dx and $d\mu_e/dx$ are less restrictive for small values of velocity than they are for large values of velocity. On the other hand, for the steady states of interest, the fraction of electrons that have large velocities is small and, therefore, large velocities are unimportant.

RANGE OF VALIDITY OF LINEAR APPROXIMATIONS

Conditions on dT_e/dx and $d\mu_e/dx$

In general, the conditions for the validity of the linearized Boltzmann equations (Eqs. 8 and 9) are velocity dependent. For the steady states of interest, however, very few electrons have high velocities and the distribution function peaks around the average kinetic energy which is about kT_e. Accordingly, the term $m_e v_e^2/2$ in Eq. 8 can be replaced by kT_e and, therefore, neglected in comparison with the negative of the chemical potential $(-\mu_{ce})$ which is usually at least one order of magnitude larger than kT_e. Moreover, the ratio $v_{ex}/\nu(v_e)$ can be approximated by an effective free path $\hat{\lambda}_e = \hat{v}_e/\hat{\nu}_e$, namely by a ratio of an effective velocity $\hat{v}_e$ and an effective collision frequency $\hat{\nu}_e$. Thus conditions 8 and 9 become

$$\left| \frac{\hat{\lambda}_e}{T_e} \frac{dT_e}{dx} \right| << \frac{kT_e}{|\mu_{ce}|} , \tag{10}$$

and

$$\left| \frac{\hat{\lambda}_e}{\mu_{ce}} \frac{d\mu_e}{dx} \right| << \frac{kT_e}{|\mu_{ce}|} . \tag{11}$$

For conditions of operation of interest to thermionic conversion, $kT_e \approx 0.2$ ev and $|\mu_{ce}| \approx 2$ to 3 ev. For example, at the points just outside the emitter electro~~n~~de E and at the points just outside the collector electrode C the ratio $kT_e/|\mu_{ce}|$ is equal to kT_E/ϕ_E and kT_C/ϕ_C, respectively, and usually

$$\frac{kT_E}{\phi_E} \approx \frac{kT_C}{\phi_C} \approx \frac{1}{20} ,$$

where T_E and T_C are the electrode temperatures, and ϕ_E and ϕ_C the electrode work functions. From conditions 10 and 11 it follows that the linear approximations are valid when the fractional changes of T_e and μ_e per effective free path $\hat{\lambda}_e$ are smaller than about 1/100. For a cesium vapor system about 10 effective free paths thick, $T_e \approx 2000°K$ and $\mu_{ce} \approx 2$ ev, the linear approximations are valid if the temperature and total potential changes across the vapor system are of the order of a few hundred degrees °K and a few tenths of an electron volt, respectively.

Under conditions of operation of thermionic converters, $T_E - T_C \approx 1000°K$ and, therefore, the linear approximations cannot be valid across the cesium vapor system. It is shown below that the linear approximations fail primarily in the sheath regions.

Failure of Linear Approximations in Sheath Regions

We will consider a sheath region of a cesium vapor system in the vicinity of an electrode. We will assume that the sheath can be analyzed by means of the relations resulting from the linearized Boltzmann equations. We will prove that the assumption is invalid when the electron flux density Γ_e is an appreciable fraction of the random flux density Γ_{re}; in other words, we will prove that the range of validity of the linear approximations in a sheath is controlled by the magnitude of the current flow and not by the magnitude of the electric field (gradient of motive ψ).

For present purposes, elastic collisional processes will be approximated by an effective electron-neutral hard-sphere free-path $\hat{\lambda}_e$ and, to first order, the effects of ionization collisions will be neglected. Both approximations have a negligible effect on the results.

Under these conditions and approximations, we can show that the electron energy flux density u_e is given by the relation

$$u_e = (2kT_e + \psi)\,\Gamma_e - \frac{8}{3}\,kT_e\,\Gamma_{re}\left(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx}\right) . \tag{12}$$

Consistent with the assumption about ionization collisions, to first order $du_e/dx \approx 0$ and $d\Gamma_e/dx \approx 0$. Hence, differentiating Eq. 12 with respect to x, replacing derivatives of ψ and $(\hat{\lambda}_e/T_e)(dT_e/dx)$ by $\Delta[\psi]_D/\lambda_D$ and $\Delta[(\lambda_e/T_e)(dT_e/dx)]_D/\lambda_D$, respectively, where $\Delta[\ ]_D$ denotes change in value over a distance of a Debye lenth λ_D, and keeping first order terms only we find

$$\Delta\left(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx}\right)_D \approx \frac{3}{8}\,\frac{\Gamma_e}{\Gamma_{re}}\,\frac{\Delta[\psi]_D}{kT_e} . \tag{13}$$

In a sheath, the change ~~$\Delta[\psi]_D$~~ $\Delta[\psi]_D/kT_e$ is about 3 and, therefore, the difference between the values of $(\hat{\lambda}_e/T_e)(dT_e/dx)$ at two points a distance λ_D apart is about Γ_e/Γ_{re}. This difference would satisfy the condition of validity of the linear approximations (Eq. 10) if and only if Γ_e/Γ_{re} is smaller than about 1/100. Under practical conditions of operation, however, Γ_e/Γ_{re} in a sheath region is larger than 1/100 and, therefore, the linear approximations are not valid in these regions. The failure of the linear approximations in the sheath regions introduces serious computational difficulties in the analysis of the cesium vapor system. Some of these difficulties have not yet been resolved.

Validity of Linear Approximations in Plasma Region

We will consider a cesium vapor system in a steady state with an accelerating emitter sheath(electron motive change $\Delta\psi_E$ through the emitter sheath is negative). We will show that the linear approximations are valid in the plasma region by proving that the electron temperature gradient and electron total potential gradient satisfy conditions 10 and 11, respectively, at the emitter sheath-plasma interface as well as throughout the plasma.

Approximating elastic collisional processes by a constant electron-neutral hard-sphere effective free path $\hat{\lambda}_e$ and, to a first order, neglecting the effects of ionization collisions, we can show that at the emitter sheath-plasma interface the electron flux density $\Gamma_e(1)$ and the electron energy flux density $u_e(1)$ are given by the relations

$$\Gamma_e(1) = \frac{4}{3}\left\{\Gamma_{re}\frac{\mu_{ce}}{kT_e}[(1-\frac{2kT_e}{\mu_{ce}})(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx}) - \frac{\hat{\lambda}_e}{\mu_{ce}}\frac{d\mu_e}{dx}]\right\}_1, \quad (14)$$

and

$$u_e(1) = \left\{2kT_e[(1+\frac{\psi}{2kT_e})\Gamma_e - \frac{4}{3}(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx})\Gamma_{re}]\right\}_1, \quad (15)$$

where subscript "1" denotes that all position-dependent quantities inside the bracket must be evaluated at the emitter sheath-plasma interface. On the other hand, $u_e(1)$ is also given by the approximate boundary condition

$$u_e(1) = \left\{2kT_e[(\frac{T_E}{T_e}-1)\Gamma^E_{\nu e} + (1+\frac{\psi_E}{2kT_e})\Gamma_e]\right\}_1, \quad (16)$$

where $\Gamma^E_{\nu e}$ is the electron emission flux density at the emitter, and ψ_E the motive just outside the emitter.

Eliminating $u_e(1)$ between Eqs. 15 and 16 we find that

$$[\frac{\Delta\psi_E}{2kT_e}\Gamma_e - (1-\frac{T_E}{T_e})\Gamma^E_{\nu e} + \frac{4}{3}(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx})\Gamma_{re}]_1 = 0. \quad (17)$$

For an accelerating sheath, $\Delta\psi_E < 0$ and $T_E < T_e(1)$ and, therefore, Eq. 17 indicates that $(dT_e/dx)_1 > 0$. When $(dT_e/dx)_1 > 0$, Eq. 14 indicates that

$$(\frac{d\mu_e}{dx})_1 < 0, \quad (18)$$

and that

$$(\frac{\hat{\lambda}_e}{\mu_{ce}}\frac{d\mu_e}{dx})_1 < \frac{3}{4}(\frac{\Gamma_e}{\Gamma_{re}}\frac{kT_e}{|\mu_{ce}|})_1 < \frac{3}{4}(\frac{\Gamma^E_{\nu e}}{\Gamma_{re}}\frac{kT_e}{|\mu_{ce}|})_1. \quad (19)$$

Next, eliminating $\Gamma_e(1)$ between Eqs. 14 and 17 we find that

$$\left\{[1-\frac{2kT_e}{\mu_{ce}}(1-\frac{kT_e}{\Delta\psi_E})]\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx} = \right.$$

$$\left.\frac{3}{2}[\frac{kT_e}{|\Delta\psi_E|}(1-\frac{T_E}{T_e})\frac{\Gamma^E_{\nu e}}{\Gamma_{re}}\frac{kT_e}{|\mu_{ce}|}] + \frac{\hat{\lambda}_e}{\mu_{ce}}\frac{d\mu_e}{dx}\right\}_1 \quad (20)$$

or, by virtue of relation 19,

$$(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx})_1 < \frac{3}{4}\left\{\left[\frac{1+\frac{2kT_e}{|\mu_{ce}|}(1+\frac{kT_e}{|\Delta\psi_E|})}{1+\frac{2kT_e}{|\Delta\psi_E|}(1-\frac{T_E}{T_e})}\right]^{-1}\frac{\Gamma^E_{\nu e}}{\Gamma_{re}}\frac{kT_e}{|\mu_{ce}|}\right\}_1. \quad (21)$$

In the right-hand sides of relations 19 and 21, the factors that multiply $kT_e/|\mu_{ce}|_1$ are appreciably smaller than unity. For example, numerical results for thermionic converters show that for $kT_e(1) \approx 0.2$ ev, $(kT_e/|\Delta\psi_E|)_1 \approx 0.2$ and $(\Gamma^E_{\nu e}/\Gamma_{re})_1 \approx 0.01$. It follows that both $[(\hat{\lambda}_e/T_e)(dT_e/dx)]_1$ and $[(\hat{\lambda}_e/\mu_{ce})(d\mu_e/dx)]_1$ are much smaller than $(kT_e/|\mu_{ce}|)_1$ and, therefore, that they satisfy conditions 10 and 11, respectively.

In the plasma region of the type under consideration both Γ_e/Γ_{re} and $\Delta[\psi]_D/kT_e$ are much smaller than unity, and, therefore, the difference $\Delta[\hat{\lambda}_e/T_e)(dT_e/dx)]_D$ (Eq.13) is much smaller than unity. It follows that $(\hat{\lambda}_e/T_e)(dT_e/dx)$ satisfies condition 10 throughout the plasma region since it satisfies it at the sheath-plasma interface.

Next, we note that the electron flux density Γ_e in the plasma is given by the relation

$$\Gamma_e = \frac{4}{3}\Gamma_{re}\frac{\mu_{ce}}{kT_e}[(1-\frac{2kT_e}{\mu_{ce}})(\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx}) - \frac{\hat{\lambda}_e}{\mu_{ce}}\frac{d\mu_e}{dx}] \quad (22)$$

Since, to first order, $\Delta[\Gamma_e]_D \approx 0$ it follows from Eq. 22 that

$$\Delta[\frac{\hat{\lambda}_e}{\mu_{ce}}\frac{d\mu_e}{dx}]_D \approx (1-\frac{2kT_e}{\mu_{ce}})\,\Delta\,[\frac{\hat{\lambda}_e}{T_e}\frac{dT_e}{dx}]_D \quad (23)$$

and, therefore, that $(\hat{\lambda}_e/\mu_{ce})(d\mu_e/dx)$ satisfies condition 11 throughout the plasma region since it satisfies it at the sheath-plasma interface. We conclude that the linear approximations are valid in the plasma region under practical conditions of operation of a thermionic converter.

CONCLUDING REMARK

For a cesium thermionic converter, we have shown that the linearized Boltzmann equations are valid in the plasma region and that their range of validity is controlled by the value of the current and not by the value of the electric field. These results indicate that the linearized Boltzmann equations and the corresponding hydrodynamic equations and fluxes can be used not only in analyses of plasma regions but also in analyses of regions in which the electric field is appreciable but the current is small compared with the random current, such as the transition regions between sheaths and the plasma.

REFERENCE

Wilkins, D.R., and E.P.Gyftopoulos, Journal of Applied Physics, 37, 9, pp. 3533 - 3540, (1966).