Reprinted from NUCLEAR SCIENCE AND ENGINEERING, Volume 10, No. 4, August 1961
Academic Press Inc. *Printed in U.S.A.*

15

NUCLEAR SCIENCE AND ENGINEERING: **10**, 372–376 (1961)

On the Range of Validity of Nonlinear Reactor Dynamics

ELIAS P. GYFTOPOULOS AND JACQUES DEVOOGHT*

Department of Nuclear Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts

Received January 16, 1961

Space-independent, conventional, nonlinear reactor dynamics equations are derived by a second-order perturbation method. A general criterion is derived indicating the conditions under which it is purposeful to use the nonlinear equations as such. It is shown that the range of validity of nonlinear dynamics is not unlimited and in case of large deviations from equilibrium higher order perturbation terms are required.

INTRODUCTION

Reactor dynamics are customarily described by a set of space-independent, nonlinear differential equations. These equations are derived from fundamental balance equations based on neutron transport theory, energy conservation etc., by some perturbation scheme which involves a second-order error.

The solution of these equations can be found either by linearization or by means of techniques applicable to nonliner differential equations, the latter being usually very complicated.

In view of the error involved in the derivation of the nonlinear differential equations and the difficulties associated with their solution, when they are not linearized, the important question arises as to what is their range of validity. In other words, one may wonder how much larger is the range of amplitude deviations over which the nonlinear equations yield useful information as compared to the range of the linearized equations.

The purpose of this paper is to examine in some detail the preceding question. To this effect, the reactor dynamics, space-independent, ordinary differential equations are derived by a perturbation procedure which puts in evidence the error inherent in the conventional form of nonlinear reactor dynamics. Then, this error is compared with the error introduced by linearization and a general criterion is established indicating the conditions under which it is purposeful to use the nonlinear equations as such. The general criterion is illustrated by some simple examples showing that nonlinear dynamics is useful only in a limited range between small and large deviations from equilibrium.

* *Present address:* Université des Bruxelles, Brussels, Belgium.

REACTOR DYNAMICS EQUATIONS

Consider all the balance equations for nuclear and nonnuclear variables that can be written at a point $\bar{r}$ of a high-power reactor, at any instant of time t. In general, these equations can be expressed in compact matrix form

$$\frac{\partial \bar{\psi}(\bar{r}, t)}{\partial t} = \bar{M}\bar{\psi}(\bar{r}, t) \qquad (1)$$

where

$\bar{\psi}(\bar{r}, t)$ is a vector whose components are all the dependent variables such as neutron densities in various energy groups, delayed neutron precursors, temperatures etc.

$\bar{M}$ is a matrix operator depending explicitly on $\bar{\psi}(\bar{r}, t)$ and implicitly on time.

$\bar{r}$ is the only independent variable, other than time, representing spatial and velocity space coordinates.

In view of the general definition of the matrix $\bar{M}$, it is evident that matrix Eq. (1) can represent any type of high-power reactor dynamics.

In principle, the solution of Eq. (1) can be found as follows. Consider the just critical reactor. All the dependent variables are independent of time and the matrix $\bar{M}$ resumes a value $\bar{M}_0$, where $\bar{M}_0$ is a linear and time-independent matrix operator. The eigenvectors and adjoint eigenvectors of this operator are established by solving the equations

$$\bar{M}_0\bar{\phi}_n(\bar{r}) = \omega_{n0}\bar{\phi}_n(\bar{r}) \tag{2}$$

$$\bar{M}_0{}^{+}\bar{\phi}_n{}^{+}(\bar{r}) = \omega_{n0}\bar{\phi}_n{}^{+}(\bar{r}) \tag{3}$$

where $\bar{\phi}_n(\bar{r})$, $\bar{\phi}_n{}^{+}(\bar{r})$ are eigenvector and adjoint eigenvector corresponding to the eigenvalue ω_{n0}. Assume the solution of Eq. (1) expanded in terms of the eigenvectors of the matrix operator $\bar{M}_0$, weighted by appropriate time-dependent functions. More precisely, assume that

$$\bar{\psi}(\bar{r}, t) = \sum_n \bar{A}_n(t)\bar{\psi}_n(\bar{r}, t) \tag{4}$$

$$\bar{\psi}_n(\bar{r}, t) = \bar{\phi}_n(\bar{r}) + \sum_{i \neq n} \bar{F}_n{}^{i}(t)\bar{\phi}_i(\bar{r}) \tag{5}$$

where $\bar{A}_n(t)$, $\bar{F}_n{}^{i}(t)$ are time-dependent amplitude coefficients.

Notice that when $\bar{M} \to \bar{M}_0$, then coefficients $\bar{A}_n(t)$ and $\bar{F}_n{}^{i}(t)$ reduce to the forms

$$\begin{aligned} &\bar{A}_n(t) \to \bar{A}_{n0}(t) = \bar{A}_{n0}e^{\omega_{n0}t} \qquad (\bar{A}_{n0} = \text{constant}) \\ &\bar{F}_n{}^{i}(t) \to 0 \end{aligned} \tag{6}$$

Replace Eq. (4) in Eq. (1) and find

$$\sum_n \frac{d\bar{A}_n(t)}{dt}\bar{\psi}_n(\bar{r}, t) + \sum_n \bar{A}_n(t)\frac{\partial\bar{\psi}_n(\bar{r}, t)}{\partial t} = \sum_n \bar{M}\bar{A}_n(t)\bar{\psi}_n(\bar{r}, t) \tag{7}$$

or, what is equivalent

$$\sum_n \bar{A}_n(t)\left\{\left[\bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt} - \bar{M}\right]\bar{\psi}_n(\bar{r}, t) + \frac{\partial\bar{\psi}_n(\bar{r}, t)}{\partial t}\right\} = 0 \tag{8}$$

The equivalence of Eqs. (7) and (8) results from the fact that if all the coefficients $\bar{A}_n(t)$ and all the vectors $\bar{\psi}_n(\bar{r}, t)$ are written in matrix forms $[A]$ and $[\psi]$, respectively

$$\sum_n \bar{A}_n(t)\bar{\psi}_n(\bar{r}, t) = [A][\psi]$$

$A]$ is a diagonal matrix, assumed as commutative [with $\bar{M}$ so that

$$\sum_n \bar{M}\bar{A}_n(t)\bar{\psi}_n(\bar{r}, t) = \sum_n \bar{A}_n(t)\bar{M}\bar{\psi}_n(\bar{r}, t)$$

Choose coefficients $\bar{A}_n(t)$ and $\bar{F}_n{}^{i}(t)$ such that

$$\left[\bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt} - \bar{M}\right]\bar{\psi}_n(\bar{r}, t) + \frac{\partial\bar{\psi}_n(\bar{r}, t)}{\partial t} = 0 \tag{9}$$

Equation (9) insures that expansion (4) is a solution of Eq. (1). Indeed, Eq. (9) leads to necessary equations for the determination of the amplitude coefficients $\bar{A}_n(t)$ and $\bar{F}_n{}^{i}(t)$. To see this more clearly, first, multiply Eq. (9) by $\bar{\phi}_n{}^{*}(\bar{r})$, the complex conjugate of $\bar{\phi}_n{}^{+}(\bar{r})$ and integrate over the entire volume of the reactor. Thus, find

$$\int \bar{\phi}_n{}^{*}(\bar{r})\bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt}\bar{\psi}_n(\bar{r}, t)\,d\bar{r} - \int \bar{\phi}_n{}^{*}(\bar{r})\bar{M}\bar{\psi}_n(\bar{r}, t)\,d\bar{r} = -\int \bar{\phi}_n{}^{*}(\bar{r})\frac{\partial\bar{\psi}_n(\bar{r}, t)}{\partial t}\,d\bar{r} \tag{10}$$

Use Eq. (5) and the orthogonality relationship

$$\int \bar{\phi}_n{}^{*}(\bar{r})\bar{\phi}_i(\bar{r})\,d\bar{r} = 0 \qquad i \neq n \tag{11}$$

to reduce Eq. (10) to the form

$$\bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt} = \frac{\int \bar{\phi}_n{}^{*}(\bar{r})\bar{M}\bar{\psi}_n(\bar{r}, t)}{\int \bar{\phi}_n{}^{*}(\bar{r})\bar{\phi}_n(\bar{r})\,d\bar{r}} \tag{12}$$

Second, multiply Eq. (9) by $\bar{\phi}_j{}^{*}(\bar{r})$, $j \neq n$, and integrate over the entire volume of the reactor. Thus find

$$\int \bar{\phi}_j{}^{*}(\bar{r})\bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt}\bar{\psi}_n(\bar{r}, t)\,d\bar{r} - \int \bar{\phi}_j{}^{*}(\bar{r})\bar{M}\bar{\psi}_n(\bar{r}, t)\,d\bar{r} = -\int \bar{\phi}_j{}^{*}(\bar{r})\frac{\partial\bar{\psi}_n(\bar{r}, t)}{\partial t}\,d\bar{r} \tag{13}$$

Use again Eq. (5) and the orthogonality relationship (11) and reduce Eq. (13) to

$$\frac{d\bar{F}_n{}^{j}(t)}{dt} = \frac{\int \bar{\phi}_j{}^{*}(\bar{r})\bar{M}\bar{\psi}_n(\bar{r}, t)\,d\bar{r}}{\int \bar{\phi}_j{}^{*}(\bar{r})\bar{\phi}_j(\bar{r})\,d\bar{r}} - \bar{A}_n(t)^{-1}\frac{d\bar{A}_n(t)}{dt}\bar{F}_n{}^{j}(t) \tag{14}$$

Equations (12) and (14) define the amplitude coefficients of the eigenvector expansion [Eq. (4)] of the solution of Eq. (1). Of course it is realized this expansion is not unique. However, the use of the eigenvectors $\bar{\phi}_n(\bar{r})$ has several advantages over other sets that might be used for the same purpose:

a. Each mode is weighted by its appropriate importance function and thus one avoids the difficulty involved in Henry's formulation (*1*) where every mode is given the importance of the funda-

mental. This problem has been discussed by Cohen (*2*).

b. In case of machine aided computations, when the set $\bar{\phi}_n(\bar{r})$ is truncated at $n = N$, the resulting equations are decoupled. This problem has been discussed by Kaplan (*3*).

It is appreciated that the use of Eqs. (12) and (14) as they stand is unwieldy for any practical problem. Simplifications are necessary and are discussed in the following sections.

NONLINEAR REACTOR DYNAMICS

An obvious simplification of Eqs. (12) and (14) is to use a perturbation scheme and write

$$\bar{M} = \bar{M}_0 + \delta\bar{M}$$
$$\bar{\psi}_n(\bar{r}, t) = \bar{\phi}_n(\bar{r}) + \delta\bar{\psi}_n(\bar{r}, t) \tag{15}$$

Notice that $\delta\bar{\psi}_n(\bar{r}, t)$ does not depend on $\bar{\phi}_n(\bar{r})$. Thus, Eq. (12) reduces to the form

$$\frac{d\bar{A}_n(t)}{dt} = \omega_{n0}\bar{A}_n(t) + \bar{A}_n(t)\frac{\int \bar{\phi}_n^*(\bar{r})\delta\bar{M}\bar{\phi}_n(\bar{r})\,d\bar{r}}{\int \bar{\phi}_n^*(\bar{r})\bar{\phi}_n(\bar{r})\,d\bar{r}} + \bar{\epsilon}_2 \tag{16}$$

$$\bar{\epsilon}_2 = \bar{A}_n(t)\frac{\int \bar{\phi}_n^*(\bar{r})\delta\bar{M}\delta\bar{\psi}_n(\bar{r}, t)\,d\bar{r}}{\int \bar{\phi}_n^*(\bar{r})\bar{\phi}_n(\bar{r})\,d\bar{r}} \tag{17}$$

Equation (16), with $\bar{\epsilon}_2$ neglected, constitutes a conventional form of nonlinear space-independent reactor dynamics. The physical interpretation is very simple. The time rate of change of the logarithm of the amplitude $\bar{A}_n(t)$ is equal to the sum of the eigenvalue of the unperturbed nth mode (playing the role of mode reactivity) and the reactivity introduced by the nonlinear couplings. The latter are bilinearly averaged over the entire reactor and retained only to first order.

The error in this formulation is of course $\bar{\epsilon}_2$.

LINEAR REACTOR DYNAMICS

Another simplification which can be introduced in Eq. (16) is to replace $\bar{A}_n(t)$ by $\bar{A}_{n0}(t)$ in front of the nonlinear couplings term. Thus, with $\bar{\epsilon}_2 = 0$

$$\frac{d\bar{A}_n(t)}{dt} = \omega_{n0}\bar{A}_n(t) + \bar{A}_{n0}(t)\frac{\int \bar{\phi}_n^*(\bar{r})\delta\bar{M}\bar{\phi}_n(\bar{r})\,d\bar{r}}{\int \bar{\phi}_n^*(\bar{r})\phi_n(\bar{r})\,d\bar{r}} \tag{18}$$

Equation (18) is the linearized space-independent version of reactor dynamics. The physical meaning is similar to the meaning of Eq. (16) but the form is simpler.

The error involved in the linearization of the nonlinear form is

$$\bar{\epsilon}_1 = [\bar{A}_n(t) - \bar{A}_{n0}(t)]\frac{\int \bar{\phi}_n^*(\bar{r})\delta\bar{M}\bar{\phi}_n(\bar{r})\,d\bar{r}}{\int \bar{\phi}_n^*(\bar{r})\bar{\phi}_n(\bar{r})\,d\bar{r}} \tag{19}$$

It is evident, from the preceding derivations, that the nonlinear form of reactor dynamics is more or less important as the error of linearization $\bar{\epsilon}_1$ is much greater or comparable to the error $\bar{\epsilon}_2$, inherent in the derivation of Eq. (16) (with $\bar{\epsilon}_2 = 0$). In other words, it is purposeful to consider nonlinear, space-independent nonlinear dynamics if and only if

$$\| \bar{\epsilon}_1 \| \gg \| \bar{\epsilon}_2 \| \tag{20}$$

Inequality (20) is the general criterion which needs to be implemented in each particular case in order to assess the range of validity of nonlinear dynamics over and above that of its linearized version. This is discussed in the next section where some further simplifications are introduced in the formalism for mathematical expediency.

RANGE OF VALIDITY OF NONLINEAR DYNAMICS

To get a better insight into the meaning of inequality (20), reduce the eigenvectors $\bar{\phi}_n(\bar{r})$ to scalar eigenfunctions $\phi_n(\bar{r})$. This reduction makes the conclusions to be derived easier to grasp without altering their general character.

Suppose that $n = 1$ corresponds to the largest eigenvalue ω_{10}. In particular, if the unperturbed reactor is at steady state, $\omega_{10} = 0$, $A_{10}(t) = A_{10}$, and one can write

$$0 < |\omega_{20}| < |\omega_{30}| < \cdots < |\omega_{n0}| \tag{21}$$

Some time after the reactor is perturbed only the fundamental mode has appreciable values, therefore it is meaningful to inquire into the restrictions imposed by the general criterion (20) on this mode:

$$\left|(A_1(t) - A_{10})\frac{\int \phi_1^*(\bar{r})\delta M\phi_1(\bar{r})\,d\bar{r}}{\int \phi_1^*(\bar{r})\phi_1(\bar{r})\,d\bar{r}}\right| \gg \left|A_1(t)\frac{\int \phi_1^*(\bar{r})\delta M\delta\psi_1(\bar{r}, t)\,d\bar{r}}{\int \phi_1^*(\bar{r})\phi_1(\bar{r})\,d\bar{r}}\right| \tag{22}$$

To pursue this further, get an estimate of $\delta\psi_1(\bar{r}, t)$. Consider first Eq. (14) for $n = 1$. Replace $\bar{M}$ by $M_0 + \delta M$, $\bar{\psi}_1(\bar{r}, t)$ by $\phi_1(\bar{r}) + \delta\psi_1(\bar{r}, t)$, and neglect second-order terms. Thus, find

$$\frac{dF_1^j(t)}{dt} = \left[\omega_{j0} - \frac{1}{A_1(t)} \cdot \frac{dA_1(t)}{dt}\right] F_1^j(t) + \frac{\int \phi_j^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_j^*(\bar{r})\phi_j(\bar{r})\, d\bar{r}} \quad (23)$$

The derivative of $F_1^j(t)$ can be neglected from Eq. (23) because higher harmonics decay very fast

$$\left|\frac{dF_1^j(t)}{dt}\right| \ll |\omega_{j0} F_1^j(t)| \quad (24)$$

Therefore

$$F_1^j(t) \cong -\frac{1}{\omega_{j0} - [1/A_1(t)][dA_1(t)/dt]} \cdot \frac{\int \phi_j^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_j^*(\bar{r})\phi_j(\bar{r})\, d\bar{r}} \quad (25)$$

Use Eq. (25) in conjunction with Eq. (5) to find

$$\delta\psi_1(\bar{r}, t) = -\sum_{j=2} \frac{\phi_j(\bar{r})}{\omega_{j0} - [1/A_1(t)][dA_1(t)/dt]} \cdot \frac{\int \phi_j^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_j^*(\bar{r})\phi_j(\bar{r})\, d\bar{r}} \quad (26)$$

Thus, the general criterion (22) reduces to:

$$\left|\frac{A_1(t) - A_{10}}{A_1(t)}\right| \gg \left|\sum_{j=2} \frac{1}{\omega_{j0} - [1/A_1(t)][dA_1(t)/dt]} \cdot \frac{\int \phi_1^*(\bar{r})\delta M\phi_j(\bar{r})\, d\bar{r} \int \phi_j^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_1^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r} \int \phi_j^*(\bar{r})\phi_j(\bar{r})\, d\bar{r}}\right| \quad (27)$$

Inequality (28) is to be considered, of course, when the deviation from equilibrium is large. In view of inequality (21), a necessary condition for (27) to be satisfied is

$$|\omega_{20}| \gg \left|\frac{\int \phi_1^*(\bar{r})\delta M\phi_2(\bar{r})\, d\bar{r} \int \phi_2^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_1^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r} \int \phi_2^*(\bar{r})\phi_2(\bar{r})\, d\bar{r}}\right| \quad (28)$$

or

$$|\omega_{20}| \gg |k\Delta\omega| \quad (29)$$

where

$$k = \frac{\left[\int \phi_1^*(\bar{r})\delta M\phi_2(\bar{r})\, d\bar{r} \int \phi_2^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r} \cdot \int \phi_1^*(\bar{r})\phi_1(\bar{r})\, d\bar{r}\right]}{\left[\int \phi_1^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}\right]^2 \int \phi_2^*(\bar{r})\phi_2(\bar{r})\, d\bar{r}}$$

$$\Delta\omega = \frac{\int \phi_1^*(\bar{r})\delta M\phi_1(\bar{r})\, d\bar{r}}{\int \phi_1^*(\bar{r})\phi_1(\bar{r})\, d\bar{r}}$$

It can be easily shown that k is of the order of unity. On the other hand, $\Delta\omega$ is proportional to the reactivity contributed by the nonlinear couplings while ω_{20} is a measure of the reactivity associated with the second mode. Since the discussion here is based on orders of magnitude, the restriction for the usefulness of the nonlinear form of reactor dynamics can be stated as follows:

Restriction on the range of validity of nonlinear dynamics: The nonlinear form of reactor dynamics is useful when the absolute value of the reactivity of the higher modes is much larger than the absolute reactivity contributed by the nonlinear couplings.

Physically this implies that if the higher order modes decay very rapidly, then the nonlinear couplings are the main source of error and the nonlinear form of reactor dynamics should be retained.

However, it turns out that this condition is difficult to satisfy in high-power reactors and the range of applicability of the nonlinear equations is very limited. To be specific, consider a one-group diffusion model for the reactor. The reactivity of the second mode is $\sim 3(k_\infty - 1)$. Take a typical temperature coefficient of reactivity $\alpha \sim 10^{-4}$–10^{-3} $\delta k/°C$. The reactivity introduced by temperature variations of the order of 10–100°C is $\Delta\rho \sim 10^{-3}$–10^{-1}. Inequality (29) is satisfied if

$$k_\infty - 1 \gg 10^{-3}\text{–}10^{-1} \quad (30)$$

Inequality (30) indicates that the range of validity of conventional nonlinear reactor dynamics may be seriously impaired in high-power reactors.

In fact, the very nature of the restriction on the range of validity of the nonlinear form is self-defeating. This is so because the nonlinear form is needed when the deviations from steady state are large and therefore the reactivity introduced by the nonlinear couplings high. But then condition (29) is more likely to be violated. This paradoxical result stems

from the fact that when the nonlinear couplings are increased, the second-order perturbation increases at a greater rate and offsets the advantage of the nonlinear form.

CONCLUSIONS

The space-independent nonlinear and linear forms of reactor dynamics are derived from the general balance equations. This is achieved by means of the eigenvectors of the complete coefficient matrix of the stationary reactor. The use of this set of eigenvectors has two advantages. First, it attributes to each mode its appropriate importance and second, it leads to uncoupled simultaneous equations when the complete set of eigenvectors is truncated.

A general criterion is derived for the assessment of the range of validity of the nonlinear form of space-independent reactor dynamics as compared to the corresponding linear form.

It is shown that the range of the nonlinear form is not unlimited and in particular is greatly impaired in the case of high-power reactors.

In conclusion, nonlinear space-independent reactor dynamics is applicable only in a limited range between small deviations from equilibrium, where the linearized equations yield good results, and large excursions where higher order correction terms are necessary.

REFERENCES

1. A. F. Henry, The application of reactor kinetics to the analysis of reactor experiments, *Nuclear Sci. and Eng.* **3**, 52–70 (1958).
2. E. Richard Cohen, Some topics in reactor kinetics, *Proc. 2nd Intern. Conf. Peaceful Uses Atomic Energy, Geneva.* P/629 (1958).
3. S. Kaplan, The property of finality and the analysis of problems in reactor space-time kinetics by various model expansions, *Nuclear Sci. and Eng.* **9**, 357–361 (1961).