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Applications of Geometric Theory to Nonlinear Reactor Dynamics

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Some basic theorems of the geometric theory of differential equations are reviewed, without proofs, in an attempt to clarify: (*a*) what relationship exists between the general solution of a set of nonlinear differential equations and the solution of its linear approximation and under what conditions this relationship can be used; and (*b*) how the geometric theory can be used to find properties of boundedness, stability, and periodicity of the solutions of nonlinear differential systems.

These theorems are illustrated by means of two-third order examples. The first is the xenon controlled reactor and the second a two-region reactor with two temperature coefficients of reactivity. It is shown without involved computations or any approximations that: (*a*) *Xenon controlled reactor*—when the reactivity controlled by xenon is smaller than the prompt xenon yield, the reactor power is always bounded but periodic oscillations may arise. When the reactivity controlled by xenon is greater than the prompt xenon yield the reactor power is unbounded; (*b*) *Two-region reactor*—this reactor does not admit periodic solutions. When the temperature coefficients are of opposite sign, conditions are derived for the reactor power to be bounded.

I. INTRODUCTION

Nuclear reactor dynamics can be fairly well represented by a set of first order, space-independent nonlinear differential equations with respect to time (*1*, *2*). The complete solution of this set of equations is in general a formidable, if not impossible task. However, under various simplifying assumptions explicit solutions of different reactor dynamics problems have been found and have been reported (*3–5*).

One of the most prominent simplifications that is repeatedly used is the linearization of the dynamic equations, which immediately leads to closed form solutions. Such solutions have the important property that they afford experimental verification by means of oscillation or other "small signal" tests without any hazards (*6*, *7*).

In view of the nonlinear character of the dynamic equations the justified question is often raised about the real value of the linearized or transfer function approach to the problem of reactor dynamics analysis, or, stated differently, about the connection between the "exact" solution and the one derived from the linearized model.

Mathematically speaking this question has a well-defined answer. However, there seems to be some misunderstanding in the nuclear reactor field. Conflicting and unqualified statements like "the linearized equations are a very good approximation" and "the linearized equations are an inadequate representation" appear very often in the nuclear literature.

The purpose of this communication is twofold. First, it gives a brief summary of some basic notions of the geometric theory of differential equations which unambiguously answer the previous question. This theory is well known in the mathematical literature (*8*) and various of its aspects pertaining to nuclear reactor dynamics have already been presented (*9–11*). However, it is felt that the power of the geometric theory is not yet fully appreciated. The power of the method lies in the fact that the properties of the solutions of a system of nonlinear differential equations can be visualized in terms of straightforward geometric or topological relationships which yield information about the existence of critical points, the boundedness and stability of the solutions, the existence of periodic solutions, and the gross interrelated features of the solutions (maxima, minima, directions of variation, etc.).

Second, this paper discusses the dynamic behavior of two reactors describable by third-order nonlinear differential equations by means of purely geometric methods. The first is a xenon controlled reactor. This reactor has been analyzed by Chernick (*12*) by means of numerical and "classical" procedures, but the present approach does not require lengthy computations or simplifying approximations. The second is a heterogeneous reactor with two temperature coefficients of reactivity which may have opposite signs. This problem had not been treated so far in full detail.

The investigation of these two reactors brings out the following important points:

a. It clearly indicates the elegance and simplicity of the geometric theory by means of which conditions for boundedness and stability in the large and existence of periodic solutions are established.

b. It definitely shows that the linearized model of a reactor does not necessarily contain all the information required for the large signal performance.

c. It implies that even though the solutions may be bounded or periodic under certain conditions, this does not necessarily mean that the upper bounds are tolerable.

2. GEOMETRIC THEORY OF AUTONOMOUS DIFFERENTIAL EQUATIONS

2.1 The Problem

The general problem to be considered in this section is the solution of an autonomous system of equations:

$$\frac{dx}{dt} = X(x) \tag{1}$$

where x is a column matrix or vector and $X(x)$ is a matrix (function of the vector x).

Assume that the origin of coordinates $x = 0$ is a critical point, namely, that $X(0) = 0$ and that it is possible to write

$$X(x) = X_1(x) + X_2(x) \tag{2}$$

where, in some sense, for x small, $X_2(x)$ is small in comparison with $X_1(x)$. For instance, $X(x)$ could be a power series of x with constant coefficients beginning with terms of degree m, and $X_1(x)$ could consist of the terms of degree m.

The system

$$\frac{dx}{dt} = X_1(x) \tag{3}$$

is known as the first approximation.

One may suspect that within a range $\| x \| < A$ (where $\| x \|$ the norm of x), the solution of Eq. (3) is very closely related to the solution of Eq. (1). However, very little is known about this when $X(x)$ is of a general form, beginning with powers of x greater than or equal to two. On the other hand, if the first approximation is a linear function then a lot of information about the solution of Eq. (1) can be gained from a detailed study of its first approximation. This problem has been extensively treated by Liapunov (*13*), Picard (*14*), Poincaré (*15*), and others. Their conclusions are summarized below for convenience. The presentation follows the pattern given by Lefschetz (*8*). Only the main theorems are repeated here. For the proofs the reader is referred to reference *8*.

2.2 Stability in the Small of a System with Linear First Approximation

Consider the system

$$\frac{dx}{dt} = Px + q(x) \tag{4}$$

where P is a constant matrix with characteristic roots λ_i and $q(x)$ is a matrix [function of the vector x such that $q(0) = 0$].

Theorem 1. Assume that $q(x)$ is continuous in a closed region $R(A)$ of the phase space x for $\| x \| \leqq A$ and satisfies a Lipschitz condition, namely

$$\| q(x) - q(x') \| \leqq \alpha(A) \| x - x' \| \tag{5}$$

Then, if all the characteristic roots of the matrix P have negative real parts, and $\alpha(A) \to 0$ for $A \to 0$, the solution of Eq. (4) is asymptotically stable at the origin of the phase space.

Theorem 2. Assume that $q(x)$ is a power series in the components of x, of degree $i \geqq 2$ and for $m_i =$ integer

$$\lambda_j \neq \sum_i m_i \lambda_i \qquad m_i \geqq 0 \qquad \sum_i m_i > 1 \tag{6}$$

Then, if all the characteristic roots of the matrix P have negative real parts, there is a spheroid region of $R(A)$, $\| x \| \leqq \rho$, in which the solution of Eq. (4) is asymptotically stable at the origin. If all the characteristic roots have positive real parts, the solution is unstable, and if some have positive and some negative real parts, the solution is conditionally stable.

The important conclusion that Theorems 1 and 2 bring out is that the study of the linear equation

$$\frac{dx}{dt} = Px \tag{4a}$$

yields useful information about the solution of Eq. (4) provided that one uses this information within the amplitude range for which it is established. It is exactly this range which qualifies the validity or insufficiency of the linear approximation, and one should not expect the results derived from an investigation of Eq. (4a) to have any meaning for large amplitude displacements.

2.3 Boundedness and Stability in the Large for Analytical Systems with Linear First Approximation

Consider again Eq. (4) with $q(x)$ as in Theorem 2, namely, an analytic function. The boundedness and stability of the solutions for large amplitude displacements can be inferred from a geometric interpretation of Liapunov's second method which consists in the following (*8*, *13*):

Theorem 3. Given the set of Eq. (4), where $q(x)$ is analytic, if there exists a positive definite scalar function $V(x)$ whose derivative $dV(x)/dt$ is of fixed opposite sign, in a region $R(A)$ of the phase space, the origin is stable. Furthermore, if $V(x) \to 0$ for $x \to 0$ the origin is asymptotically stable;

Theorem 4. If a scalar function $V(x)$ is defined (not necessarily definite) $\to 0$ with $x \to 0$ and such that $dV(x)/dt$ is definite and for every t large enough and $\| x \| < \eta$, no matter how small η is, $V(x)$ may take the sign of $dV(x)/dt$, then the origin is unstable.

These theorems are equivalent to Theorem 2. However, they afford a simple geometric interpretation extremely useful for the purposes of this discussion. When Theorem 3 is applicable, the loci of $V(x) = \epsilon$, for $\epsilon > 0$ and small, represent concentric ovals which tend to the origin as $\epsilon \to 0$. If $dV(x)/dt < 0$, the vector dx/dt points inward along every point of $V(x) = \epsilon$, and hence the solution $x \to 0$ asymptotically. When Theorem 4 is applicable the same vector points sometimes outward and sometimes inward. Thus, the system is manifestly unstable.

Reversing the argument, one might state that if there exists a surface surrounding the critical point, large enough to enclose all possible displacements, and such that the vector field dx/dt crosses it everywhere inwardly, then the solutions of Eq. (4) are bounded. If the critical point is stable in the sense of Liapunov, the trajectories $x(t)$ coalesce to the critical point. If the critical point is unstable, the system may admit periodic solutions.

The power and elegance of this interpretation will become more evident when the problem of boundedness and stability of the xenon controlled reactor and the dynamics of reactors with two temperature coefficients are discussed.

In summary, if the solution of the linear approximation of Eq. (4) is stable and the solution of Eq. (4) bounded, then the solution of the latter is also stable. If the solution of the first approximation is unstable this does not necessarily mean that the solution of Eq. (4) is unbounded or lacking periodic closed paths. These results are self-evident since two nonlinear systems may have identical linear approximations but different nonlinear terms, and the large amplitude behavior is determined by the latter and not the former.

2.4 Existence of Periodic Solutions of Analytical Systems with Linear First Approximation

Conditions for the existence of periodic solutions of autonomous systems have been established by Liapunov (*13*), Malkin (*16*), and Poincaré (*17*). They are based on the principle of analytic continuation and are extremely difficult to implement in any practical case. These conditions will not be presented here. The interested reader is referred to reference *8*. Suffice to note only that the existence of periodic solutions is based on some characteristic properties of the coefficient matrix of the linear approximation and boundedness of the solutions. This indicates that the linear approximation or transfer function approach is useful in determining the existence of periodic solutions but not adequate by itself.

As a substitute for the general conditions for the existence of periodic solutions, Poincaré's method of sections and Brouwer's fixed point theorem are outlined because they are most pertinent to the purposes of this communication.

Consider a closed region, topologically equivalent to a solid torus, free of critical points and such that the vector field dx/dt points inwardly at every point of the surface enclosing the region. In other words, consider a toroidal trap for trajectories $x(t)$ that lie inside it. Assume that the trajectories intersect a certain cross section S_1 of the torus without contact, that is without ever being tangent to it. This means that the vector field dx/dt intersects S_1 at points $Q, Q' \cdots$ and defines a topological mapping $Q \to Q'$ of S_1 into itself. If the cross section S_1 has a fixed point P, namely, if P is mapped into itself, then the particular trajectory that corresponds to $P \to P$ is closed and therefore periodic. This is Poincaré's method of sections.

In addition, if a simply connected section S_1 is mapped into itself by means of a continuous function, the mapping possesses at least one fixed point and consequently it admits at least one closed or periodic path. This is Brouwer's fixed point theorem (*18*).

Poincaré's method of sections and Brouwer's fixed point theorem prove very useful in the geometric analysis of nonlinear differential equations as will be emphasized in the subsequent examples.

3. DYNAMICS OF XENON-CONTROLLED REACTORS

3.1 The Model

The reactor model is the same as the one considered by Chernick (*12*) and is describable by the following set of equations:

$$\frac{d\phi}{dt} = \frac{\delta - \beta}{\tau}\phi + \sum_n \lambda_n C_n \tag{7}$$

$$\delta = \delta_0 - \frac{\sigma_x X}{c\sigma_f} \tag{8}$$

$$\frac{dC_n}{dt} = \frac{\beta_n}{\tau}\phi - \lambda_n C_n \tag{9}$$

$$\frac{dX}{dt} = y_x \sigma_f \phi - \sigma_x X\phi + \lambda_i I - \lambda_x X \tag{10}$$

$$\frac{dI}{dt} = y_i \sigma_f \phi - \lambda_i I \tag{11}$$

All symbols are defined in (*12*). The system of Eqs. (7)–(11) admits a critical point (excluding the one at the origin):

$$\phi_\infty = \frac{c\delta_0 \lambda_x}{\sigma_x(y - c\delta_0)} \qquad X_\infty = \frac{c\delta_0 \sigma_f}{\sigma_x}$$

$$I_\infty = \frac{y_i \sigma_f \phi_\infty}{\lambda_i} \qquad C_{n\infty} = \frac{\beta_n \phi_\infty}{\lambda_n \tau}$$

provided that $y = y_x + y_i > c\delta_0$. No critical point exists when $y < c\delta_0$.

If the delayed neutrons are considered only through their effects on the neutron mean lifetime (*12*) and the other variables are measured in terms of their equilibrium values, the system (7)–(11) reduces to

$$\frac{d\phi}{dt} = \omega_0[1 - X]\phi \tag{12}$$

$$\frac{dX}{dt} = \lambda_x[\alpha\phi + \beta I - \gamma\phi X - X] \tag{13}$$

$$\frac{dI}{dt} = \lambda_i[\phi - I] \tag{14}$$

where

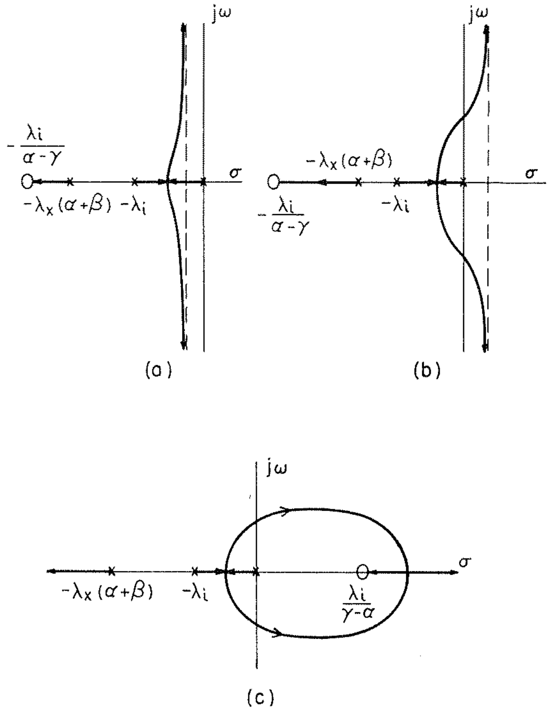

Fig. 1. Root loci of characteristic equation of the xenon controlled reactor.

$$\omega_0 = \frac{\delta_0}{\tau_e}; \qquad \tau_e = \text{equivalent neutron lifetime}$$

$$\alpha = \frac{y_x}{y - c\delta_0} \qquad \beta = \frac{y_i}{y - c\delta_0} \qquad \gamma = \frac{c\delta_0}{y - c\delta_0}$$

$$\alpha + \beta - \gamma = 1$$

3.2 Stability of the Critical Point

The type of stability at the critical point can be investigated by considering the linear approximation of Eqs. (12)–(14).

The characteristic equation of the linear approximation is

$$s^3 + [\lambda_x(\alpha + \beta) + \lambda_i]s^2 + \lambda_x[\lambda_i(\alpha + \beta) + \omega_0(\alpha - \gamma)]s + \lambda_i\lambda_x\omega_0 = 0 \tag{15}$$

or, what is equivalent

$$1 + \omega_0 \lambda_x(\alpha - \gamma)\frac{s + [\lambda_i/(\alpha - \gamma)]}{s[s + \lambda_x(\alpha + \beta)][s + \lambda_i]} = 0 \tag{16}$$

The roots of Eq. (16) can be determined by means of the root locus method. Consider two cases:

a. $\alpha > \gamma$ ($y_x > c\delta_0$). The root locus is shown in Fig. 1. The critical point is stable when the asymptote is in the right-half s plane (Fig. 1a). This is true when

$$\frac{\lambda_i}{\alpha - \gamma} \leqq \lambda_x(\alpha + \beta) + \lambda_i \tag{17}$$

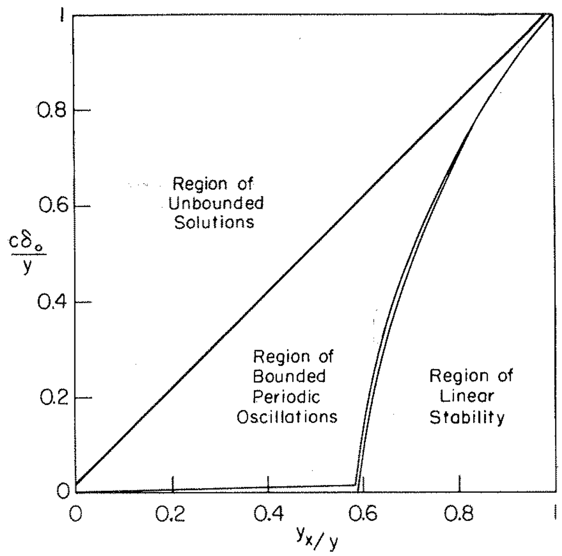

FIG. 2. Regions of linear stability, periodic oscillations and unbounded solutions of the xenon controlled reactor.

Equation (17) is fulfilled when

$$\psi = \frac{y_x}{y} \geqq \frac{\lambda_i}{\lambda_i + \lambda_x} = \frac{1}{1 + \Lambda} \quad (18)$$

$$\Delta = \frac{c\delta_0}{y} \leqq \frac{(1 + \Lambda)\psi - 1}{\psi - 1 + \Lambda} \quad (19)$$

If condition (17) is not true, the asymptote is in the left-half plane and the critical point becomes unstable when (Fig. 1b)

$$\omega^2 \geqq \lambda_i \lambda_x \frac{1}{1 - \psi + \Lambda(\beta - 1)} \quad (20)$$

$$\Delta \geqq \frac{c\lambda_i \tau_e}{y} \frac{1 + \Lambda - \Delta}{1 - (1 + \Lambda)\psi + (\psi - 1 + \Lambda)\Delta} \quad (21)$$

Equations (19) and (21) are plotted in Fig. 2 for $\lambda_x = 2.09 \times 10^{-5}\ \text{sec}^{-1}$, $\lambda_i = 2.87 \times 10^{-5}\ \text{sec}^{-1}$, $y = 6.4 \times 10^{-2}$, $\tau_e = 0.1$ sec and, $c = 1.5$. Notice that the two plots are practically identical.

b. $\alpha < \gamma$ ($y_x < c\delta_0 < y$). The root locus is shown in Fig. 1c. The conditions for instability are given again by (20) and (21). Simple inspection of Fig. 2 indicates that since the critical $\Delta < \psi$, the critical point is always unstable when $\alpha < \gamma$.

In summary, the linear approximation yields an unstable critical point when the reactivity controlled by xenon is greater than what is given by Eq. (21) [see Fig. 2] and in particular when it is greater than the prompt xenon yield. Most of these results are also given in (*12*).

3.3 Boundedness and Stability in the Large

The boundedness of the solutions for very large displacements can be found using the geometric interpretation of Liapunov's second method. More precisely, one investigates the existence of a closed surface in the phase space (ϕ, I, X) which encloses the critical point (1, 1, 1) and is intersected by the trajectories inwardly.

Consider first the case when $\alpha > \gamma$ and the surface shown in Fig. 3(a). This surface consists of eight mutually intersecting plane surfaces defined as follows:

Take the arbitrary point $A(a, a, a)$ with $a > 1$. Define the plane surfaces

ABCDFGA:	Plane $E_1 // I = 0$ through point A
ABKA:	Plane E_2 defined by $\phi - I = 0$ $(X \geqq a)$
ALKA:	Plane E_3 defined by $X = a$ $(\phi \geqq I)$
ALMGA:	Plane E_4 defined by $\phi = a$ $(a \geqq X \geqq 1, \phi \geqq I)$
BKCB:	Plane E_5 defined by $\alpha\phi + \beta I - X = -a$ $(X \leqq a, I \leqq \phi)$
CDFONKC:	Plane E_6 defined by $\phi = 0$
KLMONK:	Plane E_7 defined by $I = 0$
FGMOF:	Plane E_8 defined by $\phi = aX$ $(X \leqq 1)$

It is evident that this surface does enclose the critical point $E(1, 1, 1)$. All the trajectories cross the surface inwardly because

$\frac{dI}{dt} < 0$ on E_1

$\frac{d\phi}{dt} < 0, \quad \frac{dI}{dt} = 0$ on E_2

$\frac{dX}{dt} < 0$ on E_3 provided that a is large

$\frac{d\phi}{dt} < 0$ on E_4

$\frac{d}{dt}(\phi^2 + X^2 + I^2) < 0$ on E_5

$\phi = 0 \quad \frac{d\phi}{dt} = 0$ on E_6. The trajectories come only arbitrarily close to E_6, because $\phi < 0$ has no physical meaning

$\frac{dI}{dt} > 0$ on E_7

Finally on E_8 observe the following: The surface S

$$\alpha\phi + \beta I - \gamma\phi X - X = 0 \quad (22)$$

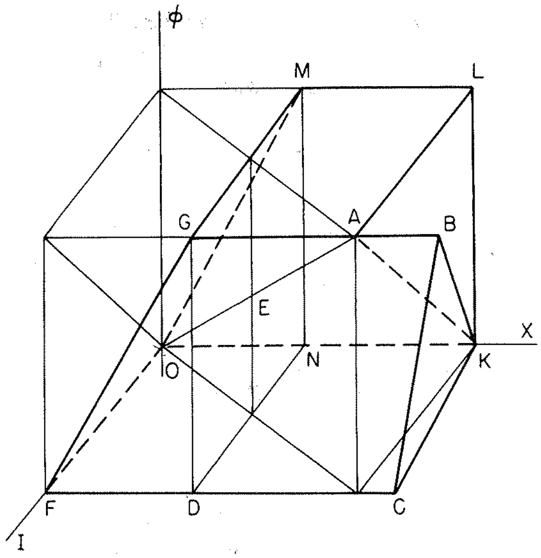

FIG. 3a

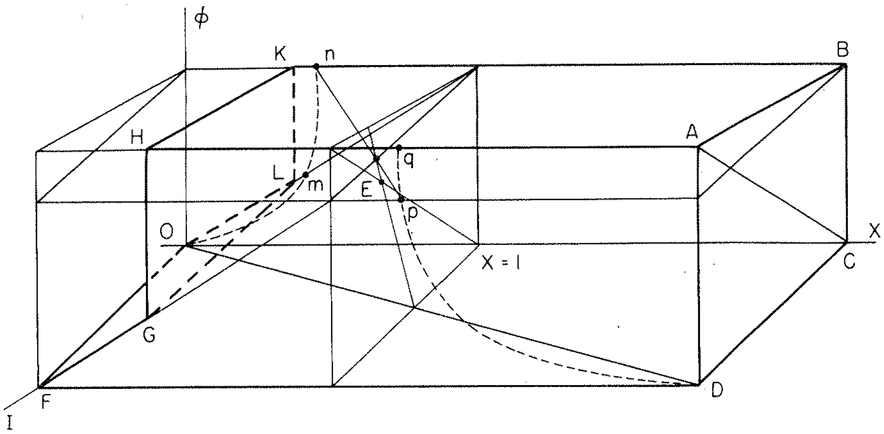

FIG. 3b

FIG. 3. (a) Closed surface surrounding critical point E and intersected inwardly by all trajectories of the xenon controlled reactor when $c\delta_0 < y_x$. (b) Closed surface surrounding critical point E and intersected inwardly by all trajectories of the xenon controlled reactor when $y_x < c\delta_0 < y_x + c\lambda_i\tau_e$.

crosses

the line: $I = 0,\ X = 1$ at $\phi = 1/(\alpha - \gamma) > 0$

the line: $\phi = I = X$ at $\phi = I = X = 1$

the line: $\phi = 0,\ X = 1$ at $I = 1/\beta > 1$

If a is chosen large, the plane E_8 lies to the left of the surface S and therefore

$$\frac{dX}{dt} > 0 \quad \text{on } E_8$$

In addition, the projection of the vector $[(d\phi/dt), (dI/dt), (dX/dt)]$ on the normal of E_8 is

$$\begin{aligned} P_n &= \omega_0(1 - X)\phi \\ &\quad - a\lambda_x[\alpha\phi + \beta I - \gamma\phi X - X] \\ &= a[(\lambda_x a\gamma - \omega_0)X^2 \\ &\quad - (\lambda_x\alpha a - \omega_0 - \lambda_x)X - \lambda_x\beta I] < 0 \end{aligned} \tag{23}$$

provided that $\alpha > \gamma$ and a large. Consequently the trajectories cross E_8 inwardly.

In view of the fact that the only requirement for the existence of the surface of Fig. 3(a) is that a be large, it is evident that such a surface can be made to include all trajectories, and since it is intersected inwardly by all trajectories, it constitutes a trajectory trap. Consequently, when $\alpha > \gamma$, the solutions of the system of Eqs. (12)–(14) are bounded. Furthermore, if the critical point is stable the solutions are asymptotically stable while if the critical point is unstable, the system admits, in general, periodic solutions as it will be shown in the next section.

Next, consider the case when $\alpha < \gamma$ and distinguish the following ranges

a. $\dfrac{\omega_0(\gamma - \alpha)}{\lambda_i \gamma} < 1$ $(y_x < c\delta_0 < y_x + c\lambda_i\tau_e)$. Consider the closed surface shown in Fig. 3(b) which consists of seven mutually intersecting plane surfaces defined as follows:

Take the arbitrary point $A(\phi = b, I = b, X = \beta b)$ with $b > 1$. Define the plane surfaces

ABCDA: Plane E_1 defined by $X = \beta b$
ADFGHA: Plane E_2 defined by $I = b$ in the region $I > \phi$
CDFOC: Plane E_3 defined by $\phi = 0$
CBKLOC: Plane E_4 defined by $I = 0$
FGLOF: Plane E_5 defined by $\phi = aX$ $(a < b, X < \alpha/\gamma)$
GHKLG: Plane E_6 defined by $X = d < \alpha/\gamma$
ABKHA: Plane E_7 defined by $\phi - [(b - a)/b]I = a$

This surface does enclose the critical point $E(1, 1, 1)$. All the trajectories cross the surface inwardly because

$$\frac{dX}{dt} < 0 \quad \text{on } E_1$$

To see this clearly consider again the ruled hyperboloid S

$$a\phi + \beta I - \gamma\phi X - X = 0 \tag{22a}$$

This hyperboloid has an asymptotic plane at $X = \alpha/\gamma < 1$ and intersects the plane $I = 0$ along the hyperbolic branch $0mn$ and the plane $I = b$ along the hyperbolic branch Dpq. For all points to the right of this surface $dX/dt < 0$ and therefore the same is true for plane E_1

$$\frac{dI}{dt} < 0 \text{ on } E_2$$

$\dfrac{d\phi}{dt} = 0, \phi = 0$ on E_3. The trajectories come only arbitrarily close to E_3 because $\phi < 0$ has no physical meaning

$$\frac{dI}{dt} > 0 \text{ on } E_4$$

The projection of the vector $[(d\phi/dt), (dI/dt), (dX/dt)]$ on the normal of plane E_5 is negative provided that a is large enough and a range of X smaller but arbitrarily close to α/γ is considered [see Eq. (23)]. Consequently, the trajectories cross E_5 inwardly.

$\dfrac{dX}{dt} > 0$ on E_6 since it is to the left of the hyperboloid S.

Finally, the projection of the vector $[(d\phi/dt), (dI/dt), (dX/dt)]$ on the normal of plane E_7 is

$$P_n' = \omega_0(1 - X)\phi - \frac{b - a}{b}\lambda_1(\phi - I) < 0 \tag{23a}$$

provided that

$$\frac{\omega_0(I - X)}{\lambda_i} < \frac{b - a}{b} < 1 \quad \text{for} \quad X > d \tag{23b}$$

In view of the fact that the range of values α, β, γ under consideration are such that

$$\frac{\omega_0(\gamma - \alpha)}{\lambda_1 \gamma} < 1$$

and d can be taken arbitrarily close to α/γ, conditions (23a, b) are readily satisfied, and the trajectories cross plane E_7 inwardly.

Since the only requirements for the existence of the surface of Fig. 3(b) are that a and b $(>a)$ be large, it can be immediately concluded that the reactor power is bounded for $c\delta_0 < y_x + c\lambda_i\tau_e$.

b. $\dfrac{\omega_0(\gamma - \alpha)}{\lambda_i \gamma} > 1$ $(c\delta_0 > y_x + c\lambda_i\tau_e)$. In this case the reactor power is unbounded because no closed surface surrounding the critical point can be found. In fact, the power diverges either monotonically or in an oscillatory manner.

Monotonic divergence is possible only when the vector $[(d\phi/dt), (dI/dt), (dX/dt)]$ has positive or zero components asymptotically. Inspection of Eqs. (12)–(14) reveals that the only possibility is

$$X = \text{constant} < 1 \qquad \frac{dX}{dt} = 0$$

The solution X = constant is admissible when the cross section of the ruled hyperboloid S [Eq. (22a)] by the plane X = constant <1 has a slope equal to the asymptotic value of $d\phi/dI$. Consequently

$$\omega_0(1 - X)\phi(\alpha - \gamma X) + \lambda_i(\phi - I)\beta = 0$$

or

$$X^2 - X\left[1 + \frac{\lambda_i}{\omega_0} + \frac{\alpha}{\gamma}\right] + \frac{\lambda_i(\alpha + \beta)}{\omega_0 \gamma} + \frac{\alpha}{\gamma} - \frac{\lambda_i}{\omega_0 \gamma}\frac{X}{\phi} = 0$$

For $\phi \to \infty$ this equation admits positive solutions smaller than unity only when

$$c\delta_0 \geqq 2\sqrt{c\lambda_i \tau_e y_i} + y_x - c\lambda_i\tau_e$$

When $y_x + c\lambda_i\tau_e < c\delta_0 < 2\sqrt{c\lambda_i \tau_e y_i} + y_x - c\lambda_i\tau_e$ monotonic divergence is not consistent with the set of Eqs. (12)–(14), and all variables ϕ, X, I diverge in an oscillatory manner.

It should be emphasized that all the previous results have been derived without any approximations or tedious computations, as opposed to other approaches to the problem. Furthermore, the existence of bounds does not necessarily imply that the bounds are tolerable. In fact, they may be extremely large.

3.4 Existence of Periodic Solutions

The question of existence of periodic solutions can be established by means of Poincaré's method of sections and Brouwer's fixed point theorem. To this effect investigate the existence of a toroidal region, not containing the critical point, whose bounding surface is intersected inwardly by the trajectories.

Consider first $\alpha > \gamma$. Notice that for all values of $\alpha > \gamma$, one of the characteristic roots of Eq. (16) is always real negative (Fig. 1a, b), say $-s_1 (s_1 > 0)$. This implies that no closed surface can be found in the neighborhood of the critical point which is crossed outwardly by the trajectories because, for $-s_1 < 0$, there are always two trajectories approaching the critical point. However, a small open ended cylindrical surface around the critical point, intersected outwardly by the trajectories does exist when the former is unstable. To prove this, proceed as follows.

Consider the system of Eqs. (12)–(14), and transfer the origin of the phase space to the critical point. Thus find

$$\frac{d\phi}{dt} = -\omega_0 X - \omega_0 \phi X \quad (24)$$

$$\frac{dX}{dt} = \lambda_x [(\alpha - \gamma)\phi + \beta I - (\gamma + 1)X - \gamma\phi X] \quad (25)$$

$$\frac{dI}{dt} = \lambda_i [\phi - I] \quad (26)$$

If the critical point is unstable, the characteristic roots of the linear approximation are, in general

$$s = -s_1 \qquad s = u \pm jv \qquad s_1, u, v > 0 \quad (27)$$

A linear transformation of ϕ, I, X into ϕ_1, I_1, X_1, by means of the modal matrix that corresponds to the characteristic roots, reduces the linear approximation to its normal form and Eqs. (24)–(26) to

$$\frac{d\phi_1}{dt} = -s_1 \phi_1 + f_1 \quad (28)$$

$$\frac{dX_1}{dt} = uX_1 + vI_1 + f_2 \quad (29)$$

$$\frac{dI_1}{dt} = -vX_1 + uI_1 + f_3 \quad (30)$$

where f_i second-order polynomials in (ϕ_1, I_1, X_1).

Define the cylinder

$$C = X_1^2 + I_1^2 > 0 \quad (31)$$

Notice that

$$\frac{dC}{dt} = 2uX_1^2 + 2uI_1^2 + 2(X_1 f_2 + I_1 f_3) > 0 \quad (32)$$

because $X_1 f_2 + I_1 f_3$ is of third order in ϕ_1, I_1, X_1 and for sufficiently small values of the latter, the first two terms in the right-hand side of Eq. (32) dominate. The meaning of Eqs. (31) and (32) is that there is a small neighborhood around the critical point in which the cylinder C is intersected outwardly by the trajectories.

The direction of the axis of the cylinder is determined by the directional cosines with respect to ϕ, I, X of the principal axis ϕ_1, which corresponds to the characteristic root $-s_1$. These cosines are

$$\cos \alpha_\phi = \frac{[s_1 - \lambda_x(\alpha + \beta)][s_1 - \lambda_i]}{\sqrt{\begin{array}{l}[s_1 - \lambda_x(\alpha + \beta)]^2 [s_1 - \lambda_i]^2 \\ \quad + \lambda_i^2 [s_1 - \lambda_x(\alpha + \beta)]^2 \\ \quad\quad + \lambda_x^2 [\lambda_i - s_1(\alpha - \gamma)]^2\end{array}}} = \frac{[s_1 - \lambda_x(\alpha + \beta)]\,[s_1 - \lambda_i]}{D} \quad (33)$$

$$\cos \alpha_I = \frac{-\lambda_i [s_1 - \lambda_x(\alpha + \beta)]}{D}$$

$$\cos \alpha_x = \frac{\lambda_x [\lambda_i - s_1(\alpha - \gamma)]}{D}$$

From Fig. 1(b) it is evident that when the critical point is unstable

$$\lambda_x(\alpha + \beta) < s_1 < [\lambda_i/(\alpha - \gamma)] \quad (34)$$

Therefore, the principal axis ϕ_1 and the cylinder are oriented as shown in Fig. 4.

Next, extend the cylinder by two funnel-like surfaces beyond the equilibrium point as shown in Fig. 4. The funnels consist of three mutually intersecting planes

EPQE and EP_1Q_1E: Plane E_9 defined by $\phi = I$

EQRE and EQ_1R_1E: Plane E_{10} defined by $bI + X = b + 1$ $(b > 0)$

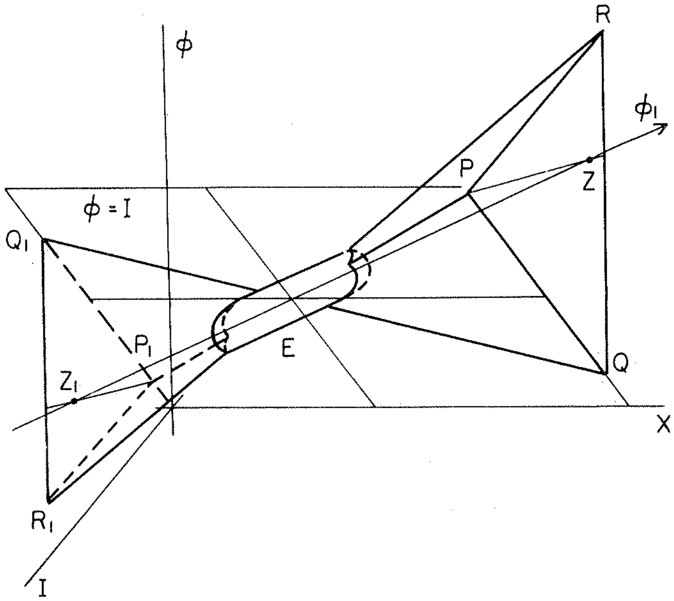

FIG. 4. Open ended surface, around critical point E intersected outwardly by all trajectories of the xenon controlled reactor—critical point unstable.

EPRE and EP_1R_1E: Plane E_{11} defined by $-c\phi + X = 1 - c$ $(c > 0)$.

Require that the slopes of planes E_{10} and E_{11} be such that the principal direction ϕ_1 is inside the funnels, a condition that is easily fulfilled.

Now, all trajectories cross the funnels outwardly because

$$\frac{d\phi}{dt} \gtrless 0 \text{ for } X \lessgtr 1 \text{ and } \frac{dI}{dt} = 0 \text{ on } E_9$$

$$\frac{dX}{dt} \gtrless 0 \text{ and } \frac{d\phi}{dt} \gtrless 0 \text{ for } X \lessgtr 1 \text{ on } E_{10}$$

$$\frac{dX}{dt} \gtrless 0 \text{ and } \frac{d\phi}{dt} \gtrless 0 \text{ for } X \lessgtr 1 \text{ on } E_{11},$$

however the slope of E_{11} can be decreased as in the case of E_8 [Fig. 3(a)] to have the trajectories intersecting outwardly. This does not conflict with the requirement of the directional cosines.

Superposition of the surfaces shown in Figs. 3(a) and 4 results in the critical point free toroidal region that was sought, if the volumes of Fig. 3(a) falling into the funnels and the cylinder as well as the origin are excluded. The exclusion of the origin is straightforward because one of the characteristic roots there is positive. Simple review of the behavior of the trajectories on the planes E_1 through E_{11} and the cylinder C immediately reveals that the bounding surface of the torus is crossed by the trajectories inwardly everywhere. Therefore the topological torus constitutes a trajectory trap.

A typical cross section of the toroidal region by the plane $I = 1$ is shown in Fig. 5. Two simply connected sections S_1 and S_2 result. Observe that any trajectory that is originally in the torus is trapped there. Furthermore, it intersects the section S_1 away from the plane of the figure, along the positive I direction, $(dI/dt > 0$ on $S_1)$ and the section S_2 towards the plane of the figure along the negative I direction $(dI/dt < 0$ on $S_2)$. This implies that a trajectory starting from a point $(\phi_0, I_0 = 1, X_0)$ on S_1 moves away and cannot return to S_1 along the negative I direction, etc. Similar considerations of the signs of the vector field $[(d\phi/dt), (dI/dt), (dX/dt)]$ in the various regions of the torus lead to the over-all conclusion that the trajectories circulate around the torus. Therefore, the simply connected section S_1 is topologically mapped into itself by a continuous vector field which circulates in a region free of critical points. According to Brouwer's fixed point theorem, the mapping possesses a fixed point, or the reactor admits periodic solutions.

Similar arguments apply when $y_x < c\delta_0 < y_x + c\lambda_i\tau_e$ and it is concluded that the reactor admits periodic solutions.

In summary, when $c\delta_0 < y_x + c\lambda_i\tau_e$ and the critical point is unstable, the xenon controlled reactor oscillates. The oscillations may be sinusoidal or of the relaxation type as discussed in reference *12*. It should be pointed out that the existence of the solid

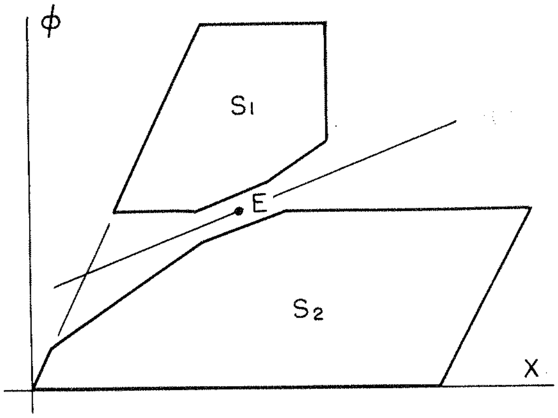

FIG. 5. Cross section of toroidal surface surrounding critical point E by the plane $I = 1$ (xenon controlled reactor).

torus is not adequate topology to guarantee either the uniqueness or the stability of the periodic solutions. Such questions can be examined by means of the general theorem of existence of periodic solutions which is beyond the scope of this communication. Also it should be noted that the existence of a negative characteristic root implies that some exceptional trajectories may indeed converge to the critical point even when the latter is unstable.

On the other hand, when $y_x + c\lambda_i\tau_e < c\delta_0$, no closed surface surrounds the critical point and the reactor power is always unbounded (Fig. 2). The unboundedness manifests itself either by diverging oscillations or by a monotonically increasing power with a bounded xenon concentration as already discussed in Section 3.3.

4. DYNAMICS OF REACTORS WITH TWO TEMPERATURE COEFFICIENTS

4.1 THE REACTOR MODEL

Two-region reactors with two temperature coefficients of reactivity have been already analyzed (*9*) by means of Liapunov's second method. However the requirement of existence of a Liapunov function may be over restricting. Here the problem is treated in all generality.

The reactor model is assumed independent of spatial coordinates and delayed neutrons are neglected. The reactor dynamics, with respect to step changes of reactivity, are describable by

$$\frac{d\phi}{dt} = \rho_1 \phi \tag{35}$$

$$\epsilon_1 \frac{dT_1'}{dt} = \eta_1 [\phi - \phi_0] - h [T_1' - T_2'] \tag{36}$$

$$\epsilon_2 \frac{dT_2'}{dt} = \eta_2 [\phi - \phi_0] + h [T_1' - T_2'] - wT_2' \tag{37}$$

$$\rho_1 = \rho_0 + r_1' T_1' + r_2' T_2' \tag{38}$$

where

- ϵ_i is heat capacity of ith region
- h is over-all heat transfer coefficient between regions (1) and (2)
- η_i is fractional power delivered to ith region ($\eta_1 + \eta_2 = 1$)
- r_1' is temperature coefficient of reactivity over neutron lifetime
- ρ_0 is step input over neutron lifetime
- T_i' is average temperature increment of ith region
- wT_2' is power removal
- ϕ is power
- ϕ_0 is steady-state power before step ρ_0 is applied.

A simple change of variable

$$T_i = T_1' + b_i T_2' \tag{39}$$

where

$$b_{1,2} = \frac{\frac{h}{\epsilon_1} - \frac{h}{\epsilon_2} - w \pm \sqrt{\left[\frac{h}{\epsilon_1} - \frac{h}{\epsilon_2} - w\right]^2 + 4\frac{h^2}{\epsilon_1\epsilon_2}}}{2\frac{h}{\epsilon_2}} \tag{40}$$

reduces the system of Eqs. (35)–(38) into the form

$$\frac{d\phi}{dt} = \rho_1\phi \tag{41}$$

$$\frac{dT_1}{dt} = a_1 [\phi - \phi_0] - g_1 T_1 \tag{42}$$

$$\frac{dT_2}{dt} = a_2 [\phi - \phi_0] - g_2 T_2 \tag{43}$$

$$\rho_1 = \rho_0 + r_1 T_1 + r_2 T_2 \tag{44}$$

with

$$g_i = \frac{h}{\epsilon_1} - b_i \frac{h}{\epsilon_2} \qquad a_i = \frac{\eta_1}{\epsilon_1} + b_i \frac{\eta_2}{\epsilon_2}$$

$$r_1 = \frac{r_2' - r_1' b_2}{b_1 - b_2} \qquad r_2 = \frac{r_1' b_1 - r_2'}{b_1 - b_2}.$$

The coefficients g_i are always positive. The coefficients a_i can also be assumed positive because, if a_i were not, a simple change of variable $T_i \to -T_i$ would result in a system with positive coefficients.

The system of Eqs. (41)–(44) admits a critical point

$$\phi_\infty = \phi_0 - \frac{\rho_0 g_1 g_2}{r_1\, a_1\, g_2 + r_2\, a_2\, g_1} \qquad \rho = 0$$

$$T_{1\infty} = \frac{a_1(\phi_\infty - \phi_0)}{g_1} \qquad T_{2\infty} = \frac{a_2(\phi_\infty - \phi_0)}{g_2}\,. \tag{45}$$

If the variables are measured in terms of their equilibrium values, Eqs. (41)–(44) reduce to

$$\frac{d\phi}{dt} = \rho\phi \tag{46}$$

$$\frac{dT_1}{dt} = g_1(\phi - T_1) + g_1 \frac{\phi_0}{\phi_\infty - \phi_0}(\phi - 1) \tag{47}$$

$$\frac{dT_2}{dt} = g_2(\phi - T_2) + g_2 \frac{\phi_0}{\phi_\infty - \phi_0}(\phi - 1) \tag{48}$$

$$\rho = \left[\frac{r_1 a_1}{g_1}(T_1 - 1) + \frac{r_2 a_2}{g_2} T_2 - 1)\right](\phi_\infty - \phi_0) \tag{49}$$

For the purposes of the subsequent discussion ϕ_0 is assumed equal to zero. This is done for mathematical expediency and does not involve any loss of generality.

4.2 Stability of the Critical Point

Proceeding as in the case of the xenon controlled reactor, it is found that the characteristic equation of the linear approximation of Eqs. (46)–(49) is ($\phi_0 = 0$)

$$s(s + g_1)(s + g_2) - \phi_\infty(r_1 a_1 + r_2 a_2) \cdot \left[s + \frac{r_1\, a_1 g_2 + r_2\, a_2 g_1}{r_1 a_1 + r_2 a_2} = 0\right] \tag{50}$$

The root loci of this equation are shown in Fig. 6(a, b, c), which indicates that

a. The critical point is stable when

$$r_1 a_1 g_1 + r_2 a_2 g_2 < 0 \qquad r_1 a_1 g_2 + r_2 a_2 g_1 < 0 \tag{51}$$

b. The critical point is conditionally stable when

$$r_1 a_1 g_1 + r_2 a_2 g_2 > 0 \qquad r_1 a_1 g_2 + r_2 a_2 g_1 < 0$$

$$g_1 > g_2 \qquad \phi_\infty < \frac{g_1/g_2(g_1 + g_2)}{r_1 a_1 g_1 + r_2 a_2 g_2} \tag{52}$$

$$\rho_0 < -\frac{(g_1 + g_2)(r_1 a_1 g_2 + r_2 a_2 g_1)}{r_1 a_1 g_1 + r_2 a_2 g_2}$$

c. No critical point exists when

$$r_1 a_1 g_2 + r_2 a_2 g_1 > 0 \tag{53}$$

In summary, when r_1, r_2 are positive, the linear approximation admits unstable solutions, when r_1, r_2 are negative, it admits stable solutions, and when $r_1 > 0$, $r_2 < 0$: if conditions (51) are satisfied, the

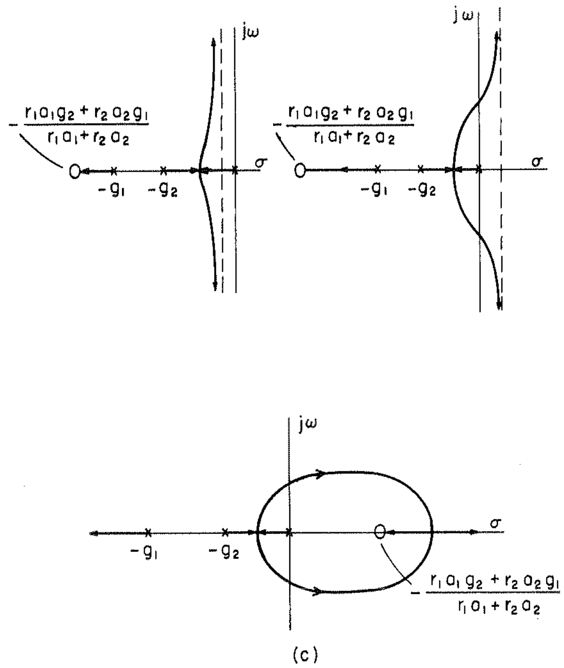

Fig. 6. Root loci of characteristic equation of two-region, two temperature coefficient of reactivity reactor.

linear approximation admits stable solutions for all values of ϕ_∞ (viz: ρ_0), while if conditions (52) are true the linear approximation admits stable solutions only for a limited range of values ϕ_∞. These results have also been presented in (*19*). In an actual case the previous conditions can of course be expressed in terms of the temperature coefficients of reactivity, etc.

4.3 Boundedness and Stability in the Large

The boundedness and stability of some solutions have already been investigated. More precisely, when r_1, $r_2 < 0$ or conditions (51) are satisfied, the reactor is asymptotically stable (*9*). However, nothing has been reported on boundedness and stability when conditions (52) are satisfied. In this case geometric theory is very helpful.

Assume $r_1 > 0$, $r_2 < 0$. Consider the phase space (ϕ, T_1, T_2) and the arbitrary point $A(b, b, b)$ with $b > 1$, shown in Fig. 7. Define a closed region by the surfaces

ABCDA:	Plane $E_1//T_1 = 0$ through point A
ABFGNA:	Plane $E_2//\phi = 0$ through point A
ADMLNA:	Plane $E_3//T_1 = 0$ through point A
BCOHFB:	Plane E_4 defined by $T_1 = 0$
KLMOHK:	Plane E_5 defined by $T_2 = 0$
CDMOC:	Plane E_6 defined by $\phi = 0$
KLNGK:	Ruled paraboloid E_7 defined by

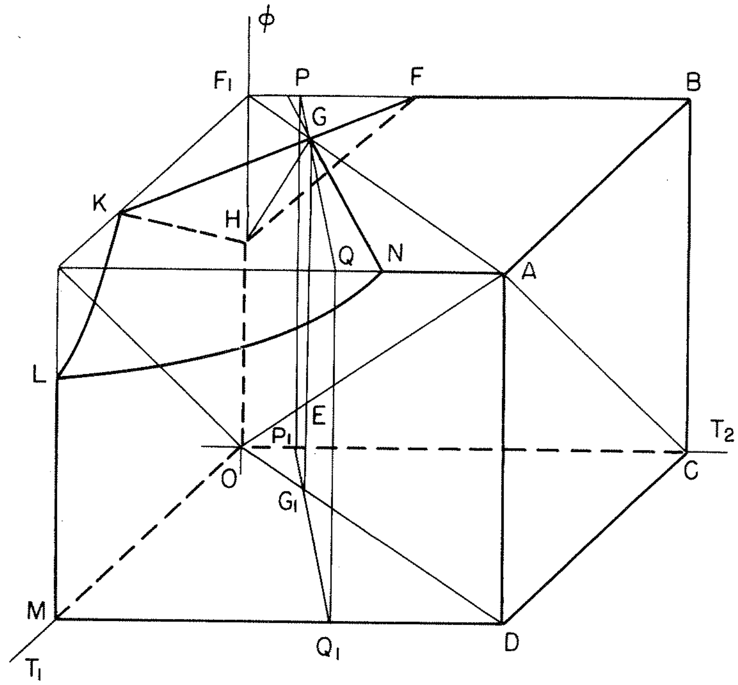

FIG. 7. Closed surface surrounding critical point E and intersected inwardly by all trajectories of the two-region reactor

$$L_1 = b - 1 - \ln b = \Big[\phi - 1 - \ln \phi - \frac{r_1 a_1}{2g_1^2}(T_1 - 1)^2 - \frac{r_2 a_2}{2g_2^2}(T_2 - 1)^2\Big] \quad (54)$$

FGKHF: Plane P_8 to be determined.

Notice that this region does enclose the critical point and is intersected by the trajectories inwardly because

$\frac{dT_2}{dt} < 0$ on E_1

$\frac{d\phi}{dt} < 0$ on E_2 because the boundaries GN and GF of this plane are to the right of the plane $PGQQ_1G_1P_1P$ defined by

$$\frac{r_1 a_1}{g_1}(T_1 - 1) + \frac{r_2 a_2}{g_2}(T_2 - 1) = 0 \quad (55)$$

$\frac{dT_1}{dt} < 0$ on E_3

$\frac{dT_1}{dt} > 0$ on E_4

$\frac{dT_2}{dt} > 0$ on E_5

$\phi = 0, \frac{d\phi}{dt} = 0$ on E_6. The trajectories come only arbitrarily close to $\phi = 0_+$, because $\phi < 0$ has no physical meaning.

The ruled paraboloid E_7 is intersected inwardly if the following conditions are satisfied:

a. L_1 be positive. Since $\phi - 1 - \ln \phi$ is always positive, this is fulfilled if

$$\left|\frac{T_1 - 1}{T_2 - 1}\right| < \sqrt{-\frac{r_2 a_2 g_1^2}{r_1 a_1 g_2^2}} \quad (56)$$

b. dL_1/dt be negative. It can be easily shown that

$$\frac{dL_1}{dt} = \frac{r_1 a_1}{g_1}(T_1 - 1)^2 + \frac{r_2 a_2}{g_2}(T_2 - 1)^2 \quad (57)$$

This is negative if

$$\left|\frac{T_1 - 1}{T_2 - 1}\right| < \sqrt{-\frac{r_2 a_2 g_1}{r_1 a_1 g_2}} \quad (58)$$

Notice that in view of conditions (52)

$$-\frac{r_2 a_2 g_1}{r_1 a_1 g_2} > \sqrt{-\frac{r_2 a_2 g_1^2}{r_1 a_1 g_2^2}} > \sqrt{-\frac{r_2 a_2 g_1}{r_1 a_1 g_2}} \quad (59)$$

provided that $r_1 a_1 + r_2 a_2 < 0$. The intersections of the paraboloid with the plane $\phi = b$ are the lines

$$\left|\frac{T_1 - 1}{T_2 - 1}\right| = \sqrt{-\frac{r_2 a_2 g_1^2}{r_1 a_1 g_2^2}} \quad (60)$$

Consequently conditions (56) and (58) are readily satisfied and furthermore GN and GF do indeed lie to the right of plane $PGQQ_1G_1P_1P$ [see inequalities (59)].

Finally, for b sufficiently large, the slope of plane E_8 can be adjusted so that the trajectories cross it inwardly. Indeed, the directional cosine with respect to ϕ, of the vector field $(d\phi/dt, dT_1/dt, dT_2/dt)$ for large b is

$$\cos \alpha_\phi \cong \frac{\rho}{\sqrt{\rho^2 + g_1^2 + g_2^2}} \quad (61)$$

and has a maximum value when ρ is evaluated at the point

$$K(\phi = b,\ T_1 = 1 + \sqrt{-r_2a_2g_1^2/r_1a_1g_2^2},\ T_2 = 0).$$

If the plane E_8 forms an angle with the ϕ-axis smaller than the one corresponding to $\cos \alpha_{\phi\max}$, then E_8 is crossed everywhere inwardly by the trajectories.

This completes the determination of the closed region. Taking b as large as desired, all trajectories can be included in the region and cannot escape from it. That is, all solutions are bounded.

In conclusion, when $r_1 > 0$, $r_2 < 0$, and $r_1a_1g_2 + r_2a_2g_1 < 0$, $r_1a_1 + r_2a_2 < 0$, the solutions of the system (46)–(49) [with $\phi_0 = 0$] are bounded, regardless of whether the critical point is stable or not. In fact, when the critical point is stable, the two region reactor is asymptotically stable, and when the critical point is unstable, periodic solutions may exist. The latter problem is discussed in the next section.

When $r_1a_1 + r_2a_2 > 0$, it can be easily shown that no closed surface can be found which is intersected everywhere inwardly by the trajectories. Consequently, the solutions are unbounded. It is interesting to note that both in the two-region reactor and in the xenon controlled reactor, when the critical point is unstable and the characteristic loci are as shown in Figs. 1(c) and 6(c), the reactors are unstable in the large. This problem seems to be related to the structural stability of third-order nonlinear systems and will be discussed in a future communication.

4.4 Existence of Periodic Solutions

The existence of periodic solutions, when the reactor is bounded in the large, is again investigated by means of Poincaré's method of sections.

Notice again that one of the characteristic roots is always real and negative, say $-s_1(s_1 > 0)$. Clearly, the existence of a toroidal region free of critical points is to be investigated.

Consider the system of Eqs. (46)–(49), and transfer the origin to the critical point. Thus find

$$\frac{d\phi}{dt} = \phi_\infty \left[\frac{r_1a_1}{g_1} T_1 + \frac{r_2a_2}{g_2} T_2\right](\phi + 1) \quad (62)$$

$$\frac{dT_1}{dt} = g_1(\phi - T_1) \quad (63)$$

$$\frac{dT_2}{dt} = g_2(\phi - T_2) \quad (64)$$

When the critical point is unstable, the characteristic roots are

$$s = -s_1 \qquad s = u \pm jv \qquad s_1,\ u,\ v > 0 \quad (65)$$

The directional cosines of the principal direction corresponding to the characteristic root $-s_1$ are

$$\cos \alpha_\phi = \frac{(s_1 - g_1)(s_1 - g_2)}{\sqrt{(s_1 - g_1)^2(s_1 - g_2)^2 + g_1^2(s_1 - g_2)^2 + g_2^2(s_1 - g_1)^2}} = \frac{(s_1 - g_1)(s_1 - g_2)}{D} \quad (66)$$

$$\cos \alpha_{T_1} = -\frac{g_1(s_1 - g_2)}{D}$$

$$\cos \alpha_{T_2} = -\frac{g_2(s_1 - g_1)}{D}$$

Notice that since $g_1 > g_2$, $(r_2a_2g_1 + r_2a_2g_1)/(r_1a_1 + r_2a_2) > s_1 > g_1 + g_2$, this direction, drawn from the critical point E, lies in the region

$$\frac{T_1}{T_2} > -\frac{r_2a_2g_1}{r_1a_1g_2} \qquad \phi > 0 \quad (67)$$

and is shown in Fig. 8 along with some typical planar cross sections of the phase space.

Plane ① ∥ to $T_1 = 0$

Plane ② ∥ to $T_2 = 0$

Plane ③ ∥ to $\phi = 0$

Plane ④ ∥ to $\frac{r_1a_1}{g_1}(T_1 - 1) + \frac{r_2a_2}{g_2}(T_2 - 1) = 0$

These cross sections indicate the general pattern of direction of the vector field at various regions of the phase space. On each planar cross section there is a dividing line (dotted line) which divides the plane into two regions intersected in opposite directions by the trajectories. The direction of crossing is indicated in the figure. For example cross section ① (plane *abcd*) is crossed along the positive T_1 direction over the region *abc* and along the negative T_1 direction along the region *acd*.

Simple inspection of the directional pattern reveals that no surface can be found which encircles

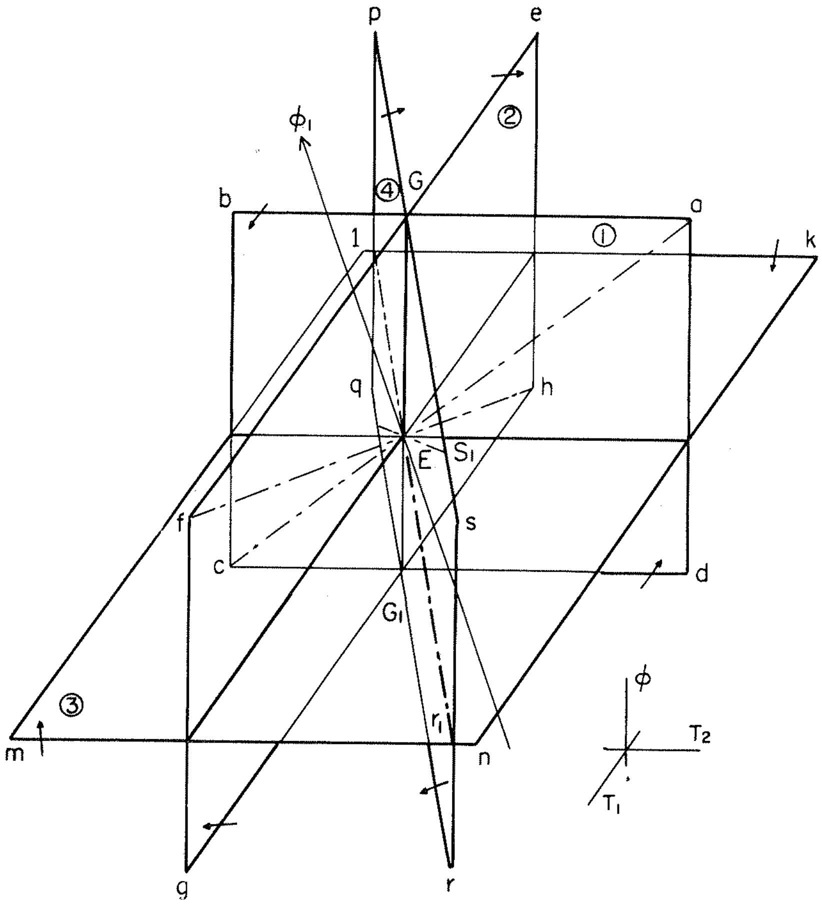

FIG. 8. Pattern of directions of trajectories of two-region reactor

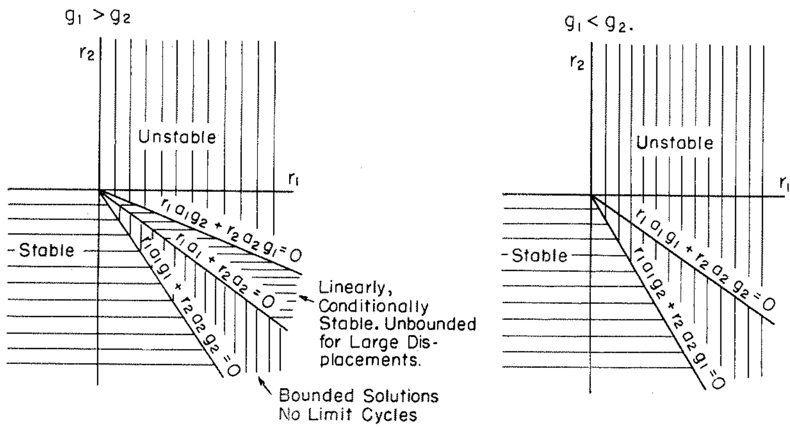

FIG. 9. Regions of stable and unstable operation for various values of the coefficients r_1, r_2 of the two-region reactor

the principal direction ϕ_1 and is crossed outwardly by the trajectories. Consequently, no toroidal surface free of critical points can be defined and no periodic solutions exist. The boundedness of the solutions implies that they eventually converge to the critical point along the principal direction ϕ_1 in spite of the fact that the critical point is unstable in the small.

In summary, when $r_1a_1 + r_2a_2 < 0$ and the critical point is unstable, the reactor variables are bounded but no periodic solutions exist. When $r_1a_1 + r_2a_2 > 0$ the solutions are unbounded. The results for the linear and nonlinear behavior of the reactor both for small and large variations are shown in Fig. 9.

It should be pointed out again that boundedness of solutions does not imply tolerable solutions.

5. CONCLUSIONS

A very brief review of the geometric theory of differential equations is given in an attempt to clarify the relationship that exists between the "small" and "large" signal behavior of nuclear reactor systems. The theory is illustrated by two specific examples, the xenon controlled reactor and a two-region reactor with two temperature coefficients of reactivity.

It is emphasized that the "small" signal or linear approximation is useful when used within a definite range of amplitude variations of the dependent variables, the magnitude of the range being defined by the nonlinear terms.

It is shown that the boundedness of the "large" signal behavior is in general dependent on the nonlinear terms, while the stability of the solutions as well as the existence of periodic oscillations depend both on the linear approximation and the nonlinear terms.

In summary, the linear or transfer function approach to reactor dynamics does not contain enough information to predict the performance of reactors when large power level changes are involved.

Both for the xenon controlled reactor and the reactor with two temperature coefficients of reactivity conditions for boundedness and stability and existence of periodic solutions are derived by simple geometric considerations and without any approximations or lengthy computations. The entire range of characteristic parameters and pertinent dependent variables is covered.

The analysis of these two reactor types clearly indicates the usefulness and elegance of the geometric theory in the field of nuclear reactor dynamics.

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